Functions of One Complex Variable

Notes and Exercises

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Part I

Exercises

Chapter 1

The Complex Number System

1.2 The field of complex numbers

1.2.1 Exercise

1. From $z\overline{z} = |z|^2$ follows $\frac{1}{z} = \frac{\overline{z}}{|z|^2} = \frac{\mathfrak{Re}(z)}{\mathfrak{Re}(z)^2+\mathfrak{Im}(z)^2} - i\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)^2+\mathfrak{Im}(z)^2}$ and $\begin{array}{rcl}\mathfrak{Re}\left(\frac{1}{z}\right) & = & \frac{\mathfrak{Re}(z)}{\mathfrak{Re}(z)^2+\mathfrak{Im}(z)^2} \\ \mathfrak{Im}\left(\frac{1}{z}\right) & = & -\frac{\mathfrak{Im}(z)}{\mathfrak{Re}(z)^2+\mathfrak{Im}(z)^2}\end{array}$

2. We have

$$\frac{z-a}{z+a} = \frac{(z-a)\overline{(z+a)}}{(z+a)\overline{(z+a)}} = \frac{z\overline{z} + (z-\overline{z})a - a^2}{z\overline{z} + (z+\overline{z})a + a^2} = \frac{|z|^2 + 2i\,\mathfrak{Im}(z)a - a^2}{|z|^2 + 2\,\mathfrak{Re}(z)a + a^2} \tag{1.1}$$

and

$$\begin{array}{rcl}\mathfrak{Re}\left(\frac{z-a}{z+a}\right) & = & \frac{|z|^2 - a^2}{|z|^2 + 2\,\mathfrak{Re}(z)a + a^2} \\ \mathfrak{Im}\left(\frac{z-a}{z+a}\right) & = & \frac{2\,\mathfrak{Im}(z)a}{|z|^2 + 2\,\mathfrak{Re}(z)a + a^2}.\end{array}$$

3. We have

$$\begin{aligned}z^3 &= (\mathfrak{Re}(z) + i\,\mathfrak{Im}(z))^3 \\ &= \mathfrak{Re}(z)^3 + 3\,\mathfrak{Re}(z)^2 i\,\mathfrak{Im}(z) + 3\,\mathfrak{Re}(z)(i\,\mathfrak{Im}(z))^2 + (i\,\mathfrak{Im}(z))^3 \\ &= \mathfrak{Re}(z)^3 - 3\,\mathfrak{Re}(z)\,\mathfrak{Im}(z)^2 + i(3\,\mathfrak{Re}(z)^2\,\mathfrak{Im}(z) - \mathfrak{Im}(z)^3) \\ &= \mathfrak{Re}(z)(\mathfrak{Re}(z)^2 - 3\,\mathfrak{Im}(z)^2) + i\,\mathfrak{Im}(z)(3\,\mathfrak{Re}(z)^2 - \mathfrak{Im}(z)^2)\end{aligned}$$

and

$$\begin{array}{rcl}\mathfrak{Re}(z^3) & = & \mathfrak{Re}(z)(\mathfrak{Re}(z)^2 - 3\,\mathfrak{Im}(z)^2) \\ \mathfrak{Im}(z^3) & = & \mathfrak{Im}(z)(3\,\mathfrak{Re}(z)^2 - \mathfrak{Im}(z)^2)\end{array}$$

4. We have

$$\frac{3+5i}{7i+1} = \frac{(3+5i)(1-7i)}{(1+7i)(1-7i)} = \frac{38-16i}{50} \tag{1.2}$$

and

$$\begin{aligned} \mathfrak{Re}\left(\frac{3+5i}{7i+1}\right) &= \frac{38}{50} \\ \mathfrak{Im}\left(\frac{3+5i}{7i+1}\right) &= -\frac{16}{50}. \end{aligned}$$

5. For point 3 we have

$$\begin{aligned} \mathfrak{Re}\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) &= -\frac{1}{2}\left(\frac{1}{4}-3\frac{3}{4}\right) = 1 \\ \mathfrak{Im}\left(\left(\frac{-1+i\sqrt{3}}{2}\right)^3\right) &= \frac{\sqrt{3}}{2}\left(3\frac{1}{4}-\frac{3}{4}\right) = 0 \end{aligned}$$

6. It is evident for point 5 that

$$\left(\frac{-1-i\sqrt{3}}{2}\right)^3 = \overline{1} = 1 \tag{1.3}$$

and so again

$$\begin{aligned} \mathfrak{Re}\left(\left(\frac{-1-i\sqrt{3}}{2}\right)^6\right) &= 1 \\ \mathfrak{Im}\left(\left(\frac{-1-i\sqrt{3}}{2}\right)^6\right) &= 0 \end{aligned}$$

7. If $n = 4k + r$, with $0 \le r < 4$, we have $i^n = i^{4k+r} = (i^4)^k i^r = 1^k i^r = i^r$, then

$$\begin{aligned} \mathfrak{Re}(i^n) &= \begin{cases} 0 & \text{if } r = 1 \text{ or } r = 3, \\ -1 & \text{if } r = 2, \\ 1 & \text{if } r = 0. \end{cases} \\ \mathfrak{Im}(i^n) &= \begin{cases} 0 & \text{if } r = 0 \text{ or } r = 2, \\ -1 & \text{if } r = 3, \\ 1 & \text{if } r = 1. \end{cases} \end{aligned}$$

8. We have

$$\begin{aligned}
\left(\frac{1+i}{\sqrt{2}}\right)^2 &= i, \\
\left(\frac{1+i}{\sqrt{2}}\right)^3 &= \left(\frac{1+i}{\sqrt{2}}\right) i = \left(\frac{-1+i}{\sqrt{2}}\right), \\
\left(\frac{1+i}{\sqrt{2}}\right)^4 &= i^2 = -1, \\
\left(\frac{1+i}{\sqrt{2}}\right)^5 &= \left(\frac{1+i}{\sqrt{2}}\right)(-1) = \left(\frac{-1-i}{\sqrt{2}}\right), \\
\left(\frac{1+i}{\sqrt{2}}\right)^6 &= i^3 = -i, \\
\left(\frac{1+i}{\sqrt{2}}\right)^7 &= \left(\frac{1+i}{\sqrt{2}}\right)(-i) = \left(\frac{1-i}{\sqrt{2}}\right), \\
\left(\frac{1+i}{\sqrt{2}}\right)^8 &= (-1)^2 = 1
\end{aligned}$$

1.2.2 Exercise

1. We have

$$|-2+i| = \sqrt{2^2+1} = \sqrt{5}, \tag{1.4}$$

$$\overline{-2+i} = -2-i. \tag{1.5}$$

2. We have

$$|-3| = 3, \tag{1.6}$$

$$\overline{-3} = -3. \tag{1.7}$$

3. We have

$$|(2+i)(4+3i)| = |2+i|\,|4+3i| = \sqrt{5}\cdot\sqrt{25} = 5\sqrt{5}, \tag{1.8}$$

$$\overline{(2+i)(4+3i)} = \overline{(2+i)}\,\overline{(4+3i)} = (2-i)(4-3i) = 5-10i. \tag{1.9}$$

4. We have

$$\left|\frac{3-i}{\sqrt{2}+3i}\right| = \frac{|3-i|}{|\sqrt{2}+3i|} = \frac{\sqrt{10}}{\sqrt{11}} = \sqrt{\frac{10}{11}}, \tag{1.10}$$

$$\overline{\left(\frac{3-i}{\sqrt{2}+3i}\right)} = \frac{3+i}{\sqrt{2}-3i} = \frac{(3+i)(\sqrt{2}+3i)}{(\sqrt{2}-3i)(\sqrt{2}+3i)} = \frac{3(\sqrt{2}-1)+i(9+\sqrt{2})}{11}. \tag{1.11}$$

5. We have

$$\left|\left(\frac{i}{i+3}\right)\right| = \frac{|i|}{|3+i|} = \frac{1}{\sqrt{10}}, \tag{1.12}$$

$$\overline{\left(\frac{i}{i+3}\right)} = \frac{\bar{i}}{\overline{3+i}} = \frac{-i}{3-i} = \frac{(-i)(3+i)}{(3-i)(3+i)} = \frac{1-3i}{10}. \tag{1.13}$$

6. We have

$$\left|(1+i)^6\right| = |1+i|^6 = \sqrt{2}^{\,6} = 2^3 = 8, \tag{1.14}$$

$$\overline{(1+i)^6} = \overline{1+i}^{\,6} = (1-i)^6 = (-2i)^3 = (-2)^3 i^3 = 8i. \tag{1.15}$$

7. We have

$$\left|i^{17}\right| = |i|^{17} = 1, \tag{1.16}$$

$$\overline{i^{17}} = \overline{i^{4\cdot 4+1}} = \overline{i^4 \cdot i^1} = \overline{1\cdot i} = \overline{i} = -i. \tag{1.17}$$

1.2.3 Exercise

- Suppose $z \in \mathbb{R}$, which in fact means $\mathfrak{Im}(z) = 0$, and $\overline{z} = z$.
- Suppose $\overline{z} = z$. Then $\mathfrak{Im}(z) = \frac{1}{2i}(z - \overline{z}) = 0$, which is the same as saying $z \in \mathbb{R}$.

1.2.4 Exercise

1. $|z+w|^2 = (z+w)\overline{(z+w)} = z\overline{z} + 2z\overline{w} + w\overline{w} = |z|^2 + z\overline{w} + \overline{(z\overline{w})} + |w|^2 = |z|^2 + 2\,\mathfrak{Re}(z\overline{w}) + |w|^2$
2. $|z-w|^2 = |z|^2 + 2\,\mathfrak{Re}(z\overline{(-w)}) + |-w|^2 = |z|^2 - 2\,\mathfrak{Re}(z\overline{w}) + |w|^2$
3. $|z+w|^2 + |z-w|^2 = |z|^2 + 2\,\mathfrak{Re}(z\overline{w}) + |w|^2 + |z|^2 - 2\,\mathfrak{Re}(z\overline{w}) + |w|^2 = 2(|z|^2 + |w|^2)$

1.2.5 Exercise

- That $|w_1 w_2| = |w_1|\,|w_2|$ is already known. Now suppose $|w_1 w_2 \cdots w_n| = |w_1|\,|w_2| \cdots |w_n|$. Then $|w_1 w_2 \cdots w_{n+1}| = |w_1 w_2 \cdots w_n|\,|w_{n+1}| =$ $|w_1|\,|w_2| \cdots |w_n|\,|w_{n+1}|$.
- That $\overline{w_1 w_2} = \overline{w_1}\,\overline{w_2}$ is already known. Now suppose $\overline{w_1 w_2 \cdots w_n} = \overline{w_1}\,\overline{w_2} \cdots \overline{w_n}$. Then $\overline{w_1 w_2 \cdots w_{n+1}} = \overline{w_1 w_2 \cdots w_n}\,\overline{w_{n+1}} =$ $\overline{w_1}\,\overline{w_2} \cdots \overline{w_n}\,\overline{w_{n+1}}$.

1.2.6 Exercise

Let

$$R(z) = \frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}, \tag{1.18}$$

where $a_i \in \mathbb{R}, 0 \le n; b_j \in \mathbb{R}, 0 \le m$. Then

$$\begin{aligned}
\overline{R(z)} &= \overline{\frac{a_0 + a_1 z + \cdots + a_n z^n}{b_0 + b_1 z + \cdots + b_m z^m}} = \\
&= \frac{\overline{a}_0 + \overline{a}_1 \overline{z} + \cdots + \overline{a}_n \overline{z}^n}{\overline{b}_0 + \overline{b}_1 \overline{z} + \cdots + \overline{b}_m \overline{z}^m} = \\
&= \frac{a_0 + a_1 \overline{z} + \cdots + a_n \overline{z}^n}{b_0 + b_1 \overline{z} + \cdots + b_m \overline{z}^m} = \\
&= R(\overline{z}).
\end{aligned}$$

1.3 The Complex Plane

1.3.1 Exercise

From

$$|z| \leq |z-w| + |w| \tag{1.19}$$

follows $|z| - |w| \leq |z-w|$, and from

$$|w| \leq |w-z| + |z| \tag{1.20}$$

follows $|w| - |z| \leq |z-w|$. So $||w| - |z|| \leq |z-w|$.
If equality holds, then either $|z| = |z-w| + |w|$ or $|w| = |w-z| + |z|$. Suppose $z \neq 0,\ w \neq 0$. In the former case, $z - w = tw$ for some $t \geq 0$, whence $z = (t+1)w$; in the latter case $w - z = uz$ for some $u \geq 0$, whence $w = (u+1)z$. In all cases, either $z = \alpha w$ for some $\alpha \geq 0$ or $w = \beta z$ for some $\beta \geq 0$.

1.3.2 Exercise

- Suppose $z_k/z_l \geq 0$ for $1 \leq k, l \leq n$ such that $z_l \neq 0$. Suppose $z_{\bar{k}} \neq 0$. In particular $z_k = \alpha_k z_{\bar{k}}$, $1 \leq k \leq n$, for some $\alpha_k \in \mathbb{R}^+$. Then

$$\begin{aligned} |z_1 + \cdots + z_n| &= |\alpha_1 z_{\bar{k}} + \cdots + \alpha_n z_{\bar{k}}| = |(\alpha_1 + \cdots + \alpha_n) z_{\bar{k}}| = \\ &= (\alpha_1 + \cdots + \alpha_n) |z_{\bar{k}}| = \alpha_1 |z_{\bar{k}}| + \cdots + \alpha_n |z_{\bar{k}}| = \\ &= |\alpha_1 z_{\bar{k}}| + \cdots + |\alpha_n z_{\bar{k}}| = |z_1| + \cdots + |z_n| \,. \end{aligned}$$

- By induction. We know that

$$|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n| \,. \tag{1.21}$$

implies $z_i/z_j \geq 0$ for $1 \leq i, j \leq n$ and $z_j \neq 0$ if $n = 2$. So suppose that if the implication holds for n, it also holds for $n+1$.
From

$$|z_1 + \cdots + z_{n+1}| = |z_1| + \cdots + |z_{n+1}| \tag{1.22}$$

we have

$$\begin{aligned} |z_1 + \cdots + z_{n+1}| = |z_1| + \cdots + |z_{n+1}| \leq \\ \leq |z_1 + \cdots + z_n| + |z_{n+1}| \leq \\ \leq |z_1| + \cdots + |z_{n+1}| \end{aligned} \tag{1.23}$$

hence

$$|z_1 + \cdots + z_n| + |z_{n+1}| = |z_1| + \cdots + |z_{n+1}| \tag{1.24}$$

and

$$|z_1 + \cdots + z_n| = |z_1| + \cdots + |z_n| \,. \tag{1.25}$$

By the inductive hypothesis, this implies $z_i/z_j \geq 0$ for $1 \leq i, j \leq n$ and $z_j \neq 0$. So if $z_{\bar{k}} \neq 0$, let $z_i = \alpha_i z_{\bar{k}}$ for $1 \leq i \leq n$, and we have

$$\begin{aligned} |\alpha_1 z_{\bar{k}} + \cdots + \alpha_n z_{\bar{k}} + z_{n+1}| &= |\alpha_1 z_{\bar{k}}| + \cdots + |\alpha_n z_{\bar{k}}| + |z_{n+1}| \\ &= \alpha_1 |z_{\bar{k}}| + \cdots + \alpha_n |z_{\bar{k}}| + |z_{n+1}| \\ &= |\alpha_1 z_{\bar{k}} + \cdots + \alpha_n z_{\bar{k}}| + |z_{n+1}| \end{aligned} \tag{1.26}$$

that is, if $\alpha = \sum_{i=1}^n \alpha_i$,

$$|\alpha z_{\bar{k}} + z_{n+1}| = |\alpha z_{\bar{k}}| + |z_{n+1}| \tag{1.27}$$

which implies $z_{n+1}/z_{\bar{k}} \geq 0$. This completes the proof that $z_i/z_j \geq 0$ for $1 \leq i, j \leq n+1$ and $z_j \neq 0$.

1.3.3 Exercise

Since the equation

$$|z-a|-|z+a|=2c \tag{1.28}$$

in the plane $\mathbb{R}^2$ says that the difference between the distances of the point $z=(x,y)$ from the points $(a,0)$ and $(-a,0)$ respectively is constant and equal to $2c$, there are three possible cases for the set

$$\mathcal{C}=\left\{(x,y)\in\mathbb{R}^2 \mid |z-a|-|z+a|=2c\right\} \tag{1.29}$$

as follows

1. $c<|a|$: $\mathcal{C}=\emptyset$.
2. $c=|a|$: $\mathcal{C}$ is the half line $x\geq a$ if $a\leq 0$ or $x\leq a$ if $a\geq 0$
3. $c>|a|$: $\mathcal{C}$ is *a branch* of the hyperbola having focuses in the points $(a,0)$ and $(-a,0)$ and axes the lines $y=\pm\sqrt{(a^2-c^2)/c^2}x$, namely the branch that encloses the point $(-a,0)$.

If a is any complex number, the set $\mathcal{C}$ is obtained from one of the former cases with the rotation that brings the point $(|a|,0)$ to the point a. For example, in the third case $\mathcal{C}$ is the branch of a hyperbola with focuses a and $-a$ that encloses the point $-a$.

1.4 Polar representation and roots of complex numbers

1.4.1 Exercise

Since $1=\text{cis}(0)$, the equation $z^n=1$ has roots

$$\text{cis}\left(\frac{2k\pi}{6}\right),\quad 0\leq k\leq 5 \tag{1.30}$$

that is

$$\begin{aligned}
z_1&=\text{cis}(0)=1\\
z_2&=\text{cis}\left(\frac{\pi}{3}\right)=\frac{1}{2}+i\frac{\sqrt{3}}{2}\\
z_3&=\text{cis}\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+i\frac{\sqrt{3}}{2}\\
z_4&=\text{cis}\left(\pi\right)=-1\\
z_5&=\text{cis}\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-i\frac{\sqrt{3}}{2}\\
z_6&=\text{cis}\left(\frac{5\pi}{3}\right)=\frac{1}{2}-i\frac{\sqrt{3}}{2}
\end{aligned}$$

1.4.2 Exercise

(a) Since $i=\text{cis}\left(\frac{\pi}{2}\right)$, the equation $z^2=i$ has roots

$$\text{cis}\left(\frac{\pi}{4}+\frac{2k\pi}{2}\right),\quad 0\leq k\leq 1 \tag{1.31}$$

that is

$$\begin{aligned}
z_1&=\text{cis}\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\\
z_2&=\text{cis}\left(\frac{5\pi}{4}\right)=-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}
\end{aligned}$$

(b) As before, the equation $z^3 = i$ has roots

$$\operatorname{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{3}\right), \quad 0 \le k \le 2 \tag{1.32}$$

that is

$$\begin{aligned} z_1 &= \operatorname{cis}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\ z_2 &= \operatorname{cis}\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \\ z_3 &= \operatorname{cis}\left(\frac{3\pi}{2}\right) = -i \end{aligned}$$

(c) Since $\left|\sqrt{3} + 3i\right| = \sqrt{12} = 2\sqrt{3}$, if $\theta = \arg(\sqrt{3} + 3i)$ it must be

$$\begin{cases} \cos\theta = \dfrac{\sqrt{3}}{2\sqrt{3}} = \dfrac{1}{2} \\ \sin\theta = \dfrac{3}{2\sqrt{3}} = \dfrac{\sqrt{3}}{2} \end{cases} \tag{1.33}$$

whence $\theta = \frac{\pi}{3}$, and the equation $z^2 = \sqrt{3} + 3i$ has roots

$$\sqrt{2\sqrt{3}}\operatorname{cis}\left(\frac{\pi}{6} + \frac{2k\pi}{2}\right), \quad 0 \le k \le 1 \tag{1.34}$$

that is

$$\begin{aligned} z_1 &= \sqrt{2\sqrt{3}}\operatorname{cis}\left(\frac{\pi}{6}\right) = \sqrt{\frac{3\sqrt{3}}{2}} + i\sqrt{\frac{\sqrt{3}}{2}} \\ z_2 &= \sqrt{2\sqrt{3}}\operatorname{cis}\left(\frac{7\pi}{6}\right) = -\sqrt{\frac{3\sqrt{3}}{2}} - i\sqrt{\frac{\sqrt{3}}{2}} \end{aligned}$$

1.4.3 Exercise

The first part is trivial:

$$(ab)^{nm} = a^{nm}b^{nm} = (a^n)^m(b^m)^n = (1)^m(1)^n = 1. \tag{1.35}$$

Of course any integer which is a multiple both of n and m would do, so the smallest value of k such that $(ab)^k = 1$ *for any given* a *and* b is the smallest common multiple $SMC(n,m)$ of n and m.
Yet that doesn't mean that, given a primitive nth root of unity and b primitive mth root of unity, $SMC(n,m)$ is the smallest integer k such that $(ab)^k = 1$. First, observe that x is a primitive rth root of unity, if and only if

$$r = \min\left\{k \in \mathbb{N}^+ \mid x^k = 1\right\} \tag{1.36}$$

and that if $x^s = 1$, then $r \mid s$. So if $(ab)^k = 1$, then

$$\begin{aligned} (ab)^{kn} &= b^{kn} = 1 \\ (ab)^{km} &= a^{km} = 1 \end{aligned}$$

which yields that $m \mid kn$ and $n \mid km$. Then every prime divisor of n or m but not of both must divide as well k. To put it more clearly, if p is a prime, $p^u \mid n$, and $p^u \nmid m$, then $p^u \mid k$; and if $p^v \mid m$, and $p^v \nmid m$, then $p^v \mid k$. So k cannot be smaller than the product of all such powers of primes. But this doesn't give a minimum. Indeed, if

$$\begin{aligned} a &= \operatorname{cis}\left(\frac{\pi}{6}\right) \\ b &= \operatorname{cis}\left(\frac{\pi}{2}\right) \end{aligned}$$

then a is a primitive 12th root of unity and b is a primitive 4th root of unity, and

$$ab = \text{cis}\left(\frac{2\pi}{3}\right) \tag{1.37}$$

so $(ab)^3 = 1$. But if

$$a = \text{cis}\left(\frac{\pi}{6}\right)$$
$$b = \text{cis}\left(\frac{2\pi}{3}\right)$$

then a is a primitive 12th root of unity and b is a primitive 3rd root of unity, but

$$ab = \text{cis}\left(\frac{5\pi}{6}\right) \tag{1.38}$$

and the smallest integer k such that $(ab)^k = 1$ is 12, not 4.
If one of a and b is nonprimitive, the said condition does not hold, for example -1 is a 2nd root of unity and also a 4th root of unity, but $(-1)(-1) = 1$.

1.4.4 Exercise

From

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$
$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

follows

$$\begin{aligned}\cos(n\theta) &= \frac{1}{2}\big((\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\big) = \\ &= \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}(i\sin\theta)^k + \right.\\ &\quad\left. + \sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}(-i\sin\theta)^k\right) = \\ &= \frac{1}{2}\left(\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}\big((i\sin\theta)^k + (-i\sin\theta)^k\big)\right) = \\ &= (\cos\theta)^n - \binom{n}{2}(\cos\theta)^{n-2}(\sin\theta)^2 + \binom{n}{4}(\cos\theta)^{n-4}(\sin\theta)^4 - \dots\end{aligned}$$

and

$$\begin{aligned}\sin(n\theta) &= \frac{1}{2i}\big((\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\big) = \\ &= \frac{1}{2i}\left(\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}(i\sin\theta)^k + \right.\\ &\quad\left. - \sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}(-i\sin\theta)^k\right) = \\ &= \frac{1}{2i}\left(\sum_{k=0}^{n}\binom{n}{k}(\cos\theta)^{n-k}\big((i\sin\theta)^k - (-i\sin\theta)^k\big)\right) = \\ &= \binom{n}{1}(\cos\theta)^{n-1}\sin\theta - \binom{n}{3}(\cos\theta)^{n-3}(\sin\theta)^3 + \dots\end{aligned}$$

1.4.5 Exercise

It is enough to know that $z^n = 1$ and $z \neq 1$, since

$$(1 + z + z^2 + \cdots + z^{n-1})(z - 1) = z^n - 1 = 0. \tag{1.39}$$

1.4.6 Exercise

Just a trivial check:

$$\varphi(t+s) = \operatorname{cis}(t+s) = \operatorname{cis}(t)\operatorname{cis}(s) = \varphi(t)\varphi(s). \tag{1.40}$$

1.4.7 Exercise

Let $z = r\operatorname{cis}\theta$, and suppose $\theta > 0$. We'll show that for some positive integer n it is $\mathfrak{Re}(z^n) < 0$.
If $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ there's nothing to prove. If $0 < \theta \leq \frac{\pi}{2}$, then there is a positive integer n such that

$$\frac{1}{n+1}\frac{\pi}{2} < \theta \leq \frac{1}{n}\frac{\pi}{2} \tag{1.41}$$

which yields

$$\frac{\pi}{2} < (n+1)\theta \leq \frac{n+1}{n}\frac{\pi}{2} \leq \pi \tag{1.42}$$

so $\mathfrak{Re}(z^{n+1}) = r^{n+1}\cos((n+1)\theta) < 0$.
If $\frac{3\pi}{2} \leq \theta < 2\pi$, let $\theta' = \theta - 2\pi$, so that $z = \operatorname{cis}\theta'$ and $-\frac{\pi}{2} \leq \theta' < 0$ and there is a positive integer n such that

$$-\frac{1}{n}\frac{\pi}{2} \leq \theta' < -\frac{1}{n+1}\frac{\pi}{2} \tag{1.43}$$

which yields

$$-\pi \leq -\frac{n+1}{n}\frac{\pi}{2} \leq \theta' < -\frac{\pi}{2} \tag{1.44}$$

and again $\mathfrak{Re}(z^{n+1}) = r^{n+1}\cos((n+1)\theta) < 0$.

1.5 Lines and half planes in the complex plane

1.5.1 Exercise

The condition is that $\operatorname{cis}\beta \perp \operatorname{cis}\alpha$, that is $\beta = \alpha + \frac{\pi}{2}$ or $\beta = \alpha + \frac{3\pi}{2}$.

1.6 The extended plane and its spherical representation

1.6.1 Exercise

Using (6.3) expressions for x_1, x_2, x_3 and x'_1, x'_2, x'_3 in (6.6) we get

$$\begin{aligned}
&d(z,z')^2 = \\
&= 2 - 2\frac{(z+\overline{z})(z'+\overline{z'}) + (-i(z-\overline{z}))(-i(z'-\overline{z'})) + (|z|^2-1)(|z'|^2-1)}{(|z|^2+1)(|z'|^2+1)} \\
&= \frac{2(z\overline{z}+1)(z'\overline{z'}+1) - 4z\overline{z'} - 4\overline{z}z' - 2z\overline{z}z'\overline{z'} + 2z\overline{z} + 2z'\overline{z'} - 2}{(|z|^2+1)(|z'|^2+1)} = \\
&= \frac{4z'\overline{z'} + 4z\overline{z} - 4z\overline{z'} - 4\overline{z}z'}{(|z|^2+1)(|z'|^2+1)} = \\
&= 4\frac{(z-z')\overline{(z-z')}}{(|z|^2+1)(|z'|^2+1)} = \\
&= 4\frac{|z-z'|^2}{(|z|^2+1)(|z'|^2+1)}.
\end{aligned}$$

If $z' = \infty$, then $x'_1 = 0$, $x'_2 = 0$ and $x'_3 = 1$, then

$$d(z,\infty)^2 = 2 - 2\frac{|z|^2-1}{|z|^2+1} = \frac{4}{|z|^2+1}. \tag{1.45}$$

1.6.2 Exercise

1. $(0, 0, 1)$
2. $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$
3. $(\frac{3}{7}, \frac{2}{7}, \frac{6}{7})$

1.6.3 Exercise

Two circumferences of maximum radius 1, lying in the coordinate planes x_1x_3 and x_2x_3 respectively.

1.6.4 Exercise

First, observe that $N \in \Lambda \iff \beta_3 = l$. Then, if the point (x_1, x_2, x_3) of $\mathbb{R}^3$ lies in the plane P, using (6.2) we get

$$\frac{2\beta_1 x + 2\beta_2 y + \beta_3(|z|^2 - 1)}{|z|^2 + 1} = l \tag{1.46}$$

and, being $|z|^2 = x^2 + y^2$,

$$(\beta_3 - l)x^2 + (\beta_3 - l)y^2 + 2\beta_1 x + 2\beta_2 y - (\beta_3 + l) = 0. \tag{1.47}$$

Now, if $N \in \Lambda$ then $\beta_3 = l$, and the former equation becomes

$$\beta_1 x + \beta_2 y - 2l = 0 \tag{1.48}$$

otherwise

$$x^2 + y^2 + \frac{2\beta_1}{\beta_3 - l}x + \frac{2\beta_2}{\beta_3 - l}y + \frac{l + \beta_3}{l - \beta_3} = 0. \tag{1.49}$$

1.6.5 Exercise

Let $X = (x_1, x_2, x_3) \in S - N$ and $Y = (y_1, y_2, y_3) \in S - N$, and let $\phi : S \to \mathbb{C}$ be the stereographic projection. Then, as we know

$$\phi(X) = \frac{x_1 + ix_2}{1 - x_3}$$
$$\phi(Y) = \frac{y_1 + iy_2}{1 - y_3}$$

whence

$$\phi(X) + \phi(Y) = \left(\frac{x_1}{1 - x_3} + \frac{y_1}{1 - y_3}\right) + i\left(\frac{x_2}{1 - x_3} + \frac{y_2}{1 - y_3}\right). \tag{1.50}$$

Let $z = x + iy = \phi(X) + \phi(Y)$, then

$$x = \frac{x_1}{1 - x_3} + \frac{y_1}{1 - y_3} = \frac{x_1(1 - y_3) + y_1(1 - x_3)}{(1 - x_3)(1 - y_3)}$$
$$y = \frac{x_2}{1 - x_3} + \frac{y_2}{1 - y_3} = \frac{x_2(1 - y_3) + y_2(1 - x_3)}{(1 - x_3)(1 - y_3)}$$

and

$$\phi^{-1}(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1}\right). \tag{1.51}$$

Now

$$\begin{aligned}
|z|^2+1 &= x^2+y^2 = \\
&= \frac{x_1^2}{(1-x_3)^2}+\frac{2x_1y_1}{(1-x_3)(1-y_3)}+\frac{y_1^2}{(1-y_3)^2}+ \\
&\quad+\frac{x_2^2}{(1-x_3)^2}+\frac{2x_2y_2}{(1-x_3)(1-y_3)}+\frac{y_2^2}{(1-y_3)^2}+1= \\
&= \frac{(x_1^2+x_2^2)(1-y_3^2)+2(x_1y_1+x_2y_2)(1-x_3)(1-y_3)+(y_1^2+y_2^2)(1-x_3^2)}{(1-x_3)^2(1-y_3)^2}+1= \\
&= \frac{(x_1^2+x_2^2)(1-y_3^2)+2(x_1y_1+x_2y_2)(1-x_3)(1-y_3)+(y_1^2+y_2^2)(1-x_3^2)}{(1-x_3)^2(1-y_3)^2}+ \\
&\quad+\frac{2(1-x_3)^2(1-y_3)^2-(1-x_3)^2(1-y_3)^2}{(1-x_3)^2(1-y_3)^2}= \\
&= \frac{(x_1^2+x_2^2+1-2x_3+x_3^2)(1-y_3^2)}{(1-x_3)^2(1-y_3)^2}+ \\
&\quad+\frac{(y_1^2+y_2^2+1-2y_3+y_3^2)(1-x_3^2)}{(1-x_3)^2(1-y_3)^2}+ \\
&\quad+\frac{2(x_1y_1+x_2y_2)(1-x_3)(1-y_3)-(1-x_3)^2(1-y_3)^2}{(1-x_3)^2(1-y_3)^2}= \\
&= \frac{2(1-x_3)(1-y_3^2)+2(1-y_3)(1-x_3^2)}{(1-x_3)^2(1-y_3)^2}+ \\
&\quad+\frac{2(x_1y_1+x_2y_2)(1-x_3)(1-y_3)-(1-x_3)^2(1-y_3)^2}{(1-x_3)^2(1-y_3)^2} \\
&= \frac{(1-x_3)(1-y_3)\left[2(1-x_3)+2(1-y_3)+2x_1y_1+2x_2y_2-(1-x_3)(1-y_3)\right]}{(1-x_3)^2(1-y_3)^2} \\
&= \frac{(1-x_3)(1-y_3)\left[3+2x_1y_1+2x_2y_2-x_3y_3-x_3-y_3\right]}{(1-x_3)^2(1-y_3)^2} \\
&= \frac{2x_1y_1+2x_2y_2-x_3y_3-x_3-y_3+3}{(1-x_3)(1-y_3)}
\end{aligned}$$

and, with similar calculations

$$|z|^2-1=\frac{2x_1y_1+2x_2y_2-3x_3y_3+x_3+y_3+1}{(1-x_3)(1-y_3)}. \tag{1.52}$$

Finally

$$\phi^{-1}(\phi(X)+\phi(Y))=\begin{pmatrix} \frac{2[x_1(1-y_3)+y_1(1-x_3)]}{2x_1y_1+2x_2y_2-x_3y_3-x_3-y_3+3} \\ \frac{2[x_2(1-y_3)+y_2(1-x_3)]}{2x_1y_1+2x_2y_2-x_3y_3-x_3-y_3+3} \\ \frac{2x_1y_1+2x_2y_2-3x_3y_3+x_3+y_3+1}{2x_1y_1+2x_2y_2-x_3y_3-x_3-y_3+3} \end{pmatrix} \tag{1.53}$$

Chapter 2

Metric Spaces and the Topology of $\mathbb{C}$

2.1 Definition and examples of metric spaces

2.1.1 Exercise

1. In both cases, that is, $\mathbb{R}$ and $\mathbb{C}$, we know that

   (a) $|z-w| \geq 0$ for all $z, w \in X$

   (b) $|z-w| = 0 \iff z-w=0 \iff z=w$ for all $z, w \in X$

   (c) $|z-w| = |-(z-w)| = |w-z|$ for all $z, w \in X$

   (d) $|z-u| = |(z-w)+(w-u)| \leq |z-w| + |w-u|$ for all $z, w, u \in X$

2. It is more correct to consider $(Y, \bar{d})$, where $\bar{d} = d_{|Y\times Y}$. Then, the conditions for $\bar{d}$ follow immediately from the ones for d.

3. Obvious.

4. The first three conditions are obvious. As for the last one, one has:

$$\begin{aligned} d(a+ib, c+id) &= \max\{|a-c|, |b-d|\} \\ d(c+id, e+if) &= \max\{|c-e|, |d-f|\} \\ d(a+ib, e+if) &= \max\{|a-e|, |b-f|\} \end{aligned}$$

   and

$$\begin{aligned} |a-e| &\leq |a-c| + |c-e| \\ |b-f| &\leq |b-d| + |d-f| \end{aligned}$$

   whence

$$\max\{|a-e|, |b-f|\} \leq \max\{|a-c|+|c-e|, |b-d|+|d-f|\} \tag{2.1}$$

   and using the easy-to-check inequality

$$\max\{x+y, u+v\} \leq \max\{x, u\} + \max\{y+v\} \tag{2.2}$$

   one gets the result.

5. The first three conditions are easy as usual. To prove the last one, one has to use *Schwarz inequality*, which will be proved later:

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leq \sqrt{\sum_{i=1}^{n} a_i^2}\sqrt{\sum_{i=1}^{n} b_i^2} \quad \text{for any } a, b \in \mathbb{R}. \tag{2.3}$$

Now, if $X = (x_1, x_2, \dots, x_n)$, $Y = (y_1, y_2, \dots, y_n)$ and $Z = (z_1, z_2, \dots, z_n)$ are three points of $\mathbb{R}^n$, one has

$$\begin{aligned}
d(X,Z)^2 &= \sum_{i=1}^{n}(x_i - z_i)^2 = \sum_{i=1}^{n}((x_i - y_i) + (y_i - z_i))^2 = \\
&= \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\sum_{i=1}^{n}(x_i - z_i)(y_i - z_i) \leq \\
&\leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + 2\left|\sum_{i=1}^{n}(x_i - z_i)(y_i - z_i)\right| \leq \\
&\leq \sum_{i=1}^{n}(x_i - y_i)^2 + \sum_{i=1}^{n}(y_i - z_i)^2 + \\
&\qquad + 2\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\sqrt{\sum_{i=1}^{n}(y_i - z_i)^2} = \\
&= \left(\sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n}(y_i - z_i)^2}\right)^2 = \\
&= \left(d(X,Y) + d(Y,Z)\right)^2 .
\end{aligned}$$

As for Schwarz inequality, if λ, a and b are any three real numbers, one has

$$\left(\lambda a - \frac{1}{\lambda}b\right)^2 = \lambda^2 a^2 + \frac{b^2}{\lambda^2} - 2ab \geq 0 \tag{2.4}$$

whence

$$2ab \leq \lambda^2 a^2 + \frac{b^2}{\lambda^2} \tag{2.5}$$

and

$$\left(\lambda a + \frac{1}{\lambda}b\right)^2 = \lambda^2 a^2 + \frac{b^2}{\lambda^2} + 2ab \geq 0 \tag{2.6}$$

whence

$$-2ab \leq \lambda^2 a^2 + \frac{b^2}{\lambda^2} \tag{2.7}$$

and from these two

$$2\,|ab| \leq \lambda^2 a^2 + \frac{b^2}{\lambda^2}. \tag{2.8}$$

So, if $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers,

$$2\left|\sum_{i=1}^{n} a_i b_i\right| \leq \lambda^2 \sum_{i=1}^{n} a_i^2 + \frac{1}{\lambda^2}\sum_{i=1}^{n} b_i^2 \tag{2.9}$$

and eventually, choosing

$$\lambda^2 = \frac{\sqrt{\sum_{i=1}^{n} b_i^2}}{\sqrt{\sum_{i=1}^{n} a_i^2}} \tag{2.10}$$

one gets the result.

2.1.2 Exercise

(a) $X = B(O,1)$ is open. If $x \in X$, let $r = 1 - d(O,x)$; clearly $r > 0$; then $B(x,r) \subseteq X$: if $y \in B(x,r)$, then $d(O,y) \leq d(O,x) + d(x,y) < d(O,x) + r = d(O,x) + 1 - d(O,x) = r$, so $y \in B(O,x)$.

(b) $X = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z = 0\}$ is closed, since $\mathbb{C} - X$ is open. If $z = x + iy \in \mathbb{C} - X$, then $B(z, |y|) \subseteq \mathbb{C} - X$: suppose for instance $y > 0$, and let $w = u + iv \in B(z, |y|)$; if $v \leq 0$, then $d(z, w) = \sqrt{(u-x)^2 + (v-y)^2} \geq \sqrt{(v-y)^2} = |v - y| = y - v \geq y$ which is impossible since $w \in B(z, |y|)$; so it must be $v > 0$ and then $w \in \mathbb{C} - X$.

(c) $X = \{z \in \mathbb{C} \mid \exists n \in \mathbb{N}^+ : z^n = 1\}$ is not open. First, observe that if $z \in X$ then $|z| = 1$. Furthermore, $1 \in X$, and in every open ball $B(1, r)$ there is an element, for instance $1 + r/2$, such that $|1 + r/2| > 1$, that is $1 + r/2 \notin X$ and $B(1, r) \nsubseteq X$. To show that X is not close, observe that $|\text{cis}(1)| = 1$ but $\text{cis}(1) \notin X$. Since we know that sin and cos are continuous functions, surely for any $\epsilon > 0$ there exists a δ such that $\sqrt{(\sin(t) - \sin(1))^2 + (\cos(t) - \cos(1))^2} < \epsilon$ whenever $|t - 1| < \delta$, that is $d(\text{cis}(t), \text{cis}(1)) < \epsilon$ if $|t - 1| < \delta$. Now, take n such that $\pi/n < \delta$ and

$$\bar{k} = \min\left\{k \in \mathbb{N} \mid 0 \leq k \leq n - 1, \frac{2k\pi}{n} > 1 - \delta\right\}. \tag{2.11}$$

Clearly,

$$\frac{2\bar{k}\pi}{n} < 1 + \delta \tag{2.12}$$

because

$$\frac{2\bar{k}\pi}{n} \geq 1 + \delta \tag{2.13}$$

yields

$$\frac{2(\bar{k}-1)\pi}{n} = \frac{2\bar{k}\pi}{n} - \frac{2\pi}{n} > \frac{2\bar{k}\pi}{n} - 2\delta \geq 1 - \delta \tag{2.14}$$

which contradicts the minimality of $\bar{k}$. So

$$\left|\frac{2\bar{k}\pi}{n} - 1\right| < \delta \tag{2.15}$$

which implies that

$$\left|\text{cis}\left(\frac{2\bar{k}\pi}{n}\right) - \text{cis}(1)\right| < \epsilon. \tag{2.16}$$

What we have proved is this: given any $\epsilon > 0$ there exists an element $z = \text{cis}(2\bar{k}\pi/n)$ in X such that $z \in B(1, \epsilon)$, so $\mathbb{C} - X$ is not open and X is not closed.

(d) $X = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z = 0, 0 \leq \mathfrak{Re}\, z < 1\}$ is not open: clearly $0 \in X$, but there is no $r > 0$ such that $B(0, r) \subseteq X$, as for instance $ir/2 \in B(0, r)$ but $ir/2 \notin X$. X is not closed: $1 \notin X$ but there is no $r > 0$ such that $B(1, r) \subseteq \mathbb{C} - X$, as for instance, if $\eta = \min\{r/2, 1/2\}$, $1 - \eta \in B(1, r)$ but $1 - \eta \notin \mathbb{C} - X$ as $1 - \eta \in X$.

(e) $X = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z = 0, 0 \leq \mathfrak{Re}\, z \leq 1\}$ is closed: $X = Y \cap Z$ where $Y = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z = 0\}$ and $Z = \{z \in \mathbb{C} \mid 0 \leq \mathfrak{Re}\, z \leq 1\}$; we have already seen that the real axis Y is closed, and it is easy to see that Z is closed too: if $z \in \mathbb{C} - Z$ and for instance $\mathfrak{Re}\, z > 1$, then $B(z, (\mathfrak{Re}\, z - 1)/2) \subseteq \mathbb{C} - Z$. Similarly if $\mathfrak{Re}\, z < 0$.

2.1.3 Exercise

The open ball $B(x, r)$ is open because, if $y \in B(x, r)$, then $B(y, r - d(x, y)) \subseteq B(x, r)$: in fact, if $z \in B(y, r - d(x, y))$, then $d(y, z) < r - d(x, y)$ and $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$ so $z \in B(x, r)$.

The closed ball $\bar{B}(x, r)$ is indeed closed because, if $y \in X - \bar{B}(x, r)$, then $B(y, d(x, y) - r) \subseteq X - \bar{B}(x, r)$: in fact, if $z \in B(y, d(x, y) - r)$ then $d(y, z) < d(x, y) - r$ and $d(x, y) \leq d(x, z) + d(z, y)$ whence $d(x, z) \geq d(x, y) - d(y, z) > d(x, y) + r - d(x, y) = r$ so $z \notin \bar{B}(x, r)$ and $z \in X - \bar{B}(x, r)$.

2.1.4 Exercise

If all the sets G_j, $j \in J$ are open and $x \in \bigcup_{j \in J} G_j$, then there is at least an index $j_0 \in J$ such that $x \in G_{j_0}$; then there is an open ball $B(x, r)$ such that $B(x, r) \subseteq G_{j_0}$ which yields $B(x, r) \subseteq \bigcup_{j \in J} G_j$.

2.1.5 Exercise

As stated, it is just a trivial and straightforward application of de Morgan's laws to Proposition 1.9

2.1.6 Exercise

Just observe that $G = X - (X - G)$.

2.1.7 Exercise

More generally: if (X, d) is a metric space and $\phi : X \to Y$ is a bijection, then one can give Y a metric space structure, defining $d' : Y \times Y \to \mathbb{R}$ by $d'(y_1, y_2) = d(\phi^{-1}(y_1), \phi^{-1}(y_2))$. It is a routine check to prove that the function d' satisfies the conditions in order to be a distance function.

2.1.8 Exercise

Let G be an open subset of X and $Y \subseteq X$. If $x \in G \cap Y$, since $x \in G$ and G is open, there is an open ball $B(x, r)$ of X such that $B(x, r) \subseteq G$; then $B(x, r) \cap Y$ is an open ball of Y and $B(x, r) \cap Y \subseteq G \cap Y$.

Let G be an open subset of Y (that is, in the topology of Y induced by the distance function d restricted to Y). Then for every $x \in G$ there is an open ball *of* Y $B_Y(x, r_x)$ such that $B_Y(x, r_x) \subseteq G$, where $B_Y(x, r_x) = B(x, r_x) \cap Y$, and $B(x, r_x) = \{y \in X \mid d(x, y) < r_x\}$. Clearly we have

$$G = Y \cap \left(\bigcup_{x \in G} B(x, r_x) \right) \tag{2.17}$$

and

$$A = \bigcup_{x \in G} B(x, r_x) \tag{2.18}$$

is an open subset of X.

2.1.9 Exercise

Let G be a closed subset of X and $Y \subseteq X$. Then $X - G$ is an open subset of X, and $(X - G) \cap Y$ is an open subset of Y, as seen in Exercise 8. Since clearly $Y - G = (X - G) \cap Y$, G is a closed subset of Y.

Let G be a closed subset of Y (that is, in the topology of Y induced by the distance function d restricted to Y). Then $Y - G$ is an open subset of Y, so, as seen in Exercise 8, there is an open subset A of X such that $Y - G = A \cap Y$ and clearly $G = (X - A) \cap Y$.

2.1.10 Exercise

Here $\ddot{A}$ will denote the interior of the set A and $\overline{A}$ the closure of the set A.

(a) If $\ddot{A} = A$, then A is open, since so is $\ddot{A}$ being union of open sets. If A is open, clearly $A \subseteq \ddot{A}$, as $A \subseteq A$. But for any set S it is true that $\ddot{S} \subseteq S$, being $\ddot{S}$ union of subsets of S.

(b) If $\overline{A} = A$, then A is closed, since so is $\overline{A}$ being intersection of closed sets. If A is closed, clearly $A \supseteq \overline{A}$, as $A \supseteq A$. But for any set S it is true that $\overline{S} \supseteq S$, being $\overline{S}$ intersection of supersets of S.

(c)

1. Since $X - \overline{(X - A)}$ is open and $X - \overline{(X - A)} \subseteq A$, we have $X - \overline{(X - A)} \subseteq \ddot{A}$. Furthermore, $X - \ddot{A}$ is closed and $X - \ddot{A} \supseteq X - A$, then $X - \ddot{A} \supseteq \overline{X - A}$ and $\ddot{A} \subseteq X - \overline{X - A}$.
2. Since $X - (X \ddot{-} A)$ is closed and $X - (X \ddot{-} A) \supseteq A$, we have $X - (X \ddot{-} A) \supseteq \overline{A}$. Furthermore, $X - \overline{A}$ is open and $X - \overline{A} \subseteq X - A$, then $X - \overline{A} \subseteq X \ddot{-} A$ and $\overline{A} \supseteq X - X \ddot{-} A$.

3. Since $\partial A = \overline{A} \cap \overline{(X-A)}$, then $\partial A \subseteq \overline{A}$ and $\partial A \subseteq \overline{(X-A)}$ and from the latter and Point 1 we get $\partial A \subseteq X - \ddot{A}$, so $\partial A \subseteq \overline{A} - \ddot{A}$.
Clearly, $\overline{A} - \ddot{A} \subseteq \overline{A}$; also, $\overline{A} - \ddot{A} \subseteq X - \ddot{A}$ so, by Point 2, $\overline{A} - \ddot{A} \subseteq \overline{(X-A)}$. Then $\overline{A} - \ddot{A} \subseteq \overline{A} \cap \overline{(X-A)} = \partial A$.

(d) Both $A \subseteq \overline{A \cup B}$ and $B \subseteq \overline{A \cup B}$ hold, so both $\overline{A} \subseteq \overline{A \cup B}$ and $\overline{B} \subseteq \overline{A \cup B}$ hold too, whence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.
Furthermore, $\overline{A} \cup \overline{B}$ is a closed subset of X and $A \cup B \subseteq \overline{A} \cup \overline{B}$, so $\overline{(A \cup B)} \subseteq \overline{A} \cup \overline{B}$.

(e) If $x \in \ddot{A}$, there exists an open set B such that $x \in B \subseteq A$, and then there is an open ball $B(x,r)$ such that $B(x,r) \subseteq B$; clearly $B(x,r) \subseteq A$.
If there is an open ball $B(x,r)$ such that $B(x,r) \subseteq A$, then $B(x,r) \subseteq \ddot{A}$ and $x \in \ddot{A}$.

2.1.11 Exercise

Here we use the well known fact that if α is an irrational number the set $\{n + m\alpha \mid n \in \mathbb{N}, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Now, if $t \in T = \{z \in \mathbb{C} \mid |z| = 1\}$, there is a real number β such that $t = \operatorname{cis}\beta$ and for any $\epsilon > 0$ there exists δ such that $|\theta - \beta| < \delta$ implies $|\operatorname{cis}\theta - \operatorname{cis}\beta| = |\operatorname{cis}\theta - t| < \epsilon$. For the said well known fact, there exist two integers n and m with $n \geq 0$ such that $|n + 2m\pi - \beta| < \delta$, which yields $|\operatorname{cis}(n + 2m\pi) - \operatorname{cis}\beta| = |\operatorname{cis} n - t| < \epsilon$. So we have proved that for any $t \in T$ and for any $\epsilon > 0$ in the open ball $B(t,\epsilon)$ lies an element of $S = \{\operatorname{cis} k \mid k \in \mathbb{N}\}$, and this for Point (f) of Proposition 1.13 means that every point t of T belongs to $\overline{S}$, that is, $T \subseteq \overline{S}$. But since T is closed and $T \supseteq S$, also $\overline{S} \subseteq T$ holds, and eventually $T = \overline{S}$.

To prove that T is closed, just observe that if $\notin T$, then $B(z, |z| - 1) \subseteq X - T$. Or better still, that $T = \bar{B}(0,1) \cap (X - B(0,1))$.

The set $S_\theta = \{\operatorname{cis} k\theta \mid k \in \mathbb{N}\}$ is dense in T if π/θ is irrational: just as before, for any $\beta \in \mathbb{R}$ and for any $\delta > 0$ there are two integers n and m with $n \geq 0$ such that $|n + 2m\pi/\theta - \beta/\theta| < \delta/|\theta|$ or $|n\theta + 2m\pi - \beta| < \delta$, then if $t = \operatorname{cis}\beta \in T$, for any $\epsilon > 0$ there are two integers n and m with $n \geq 0$ such that $|\operatorname{cis}(n\theta + 2m\pi) - \operatorname{cis}\beta| = |\operatorname{cis} n\theta - t| < \epsilon$.

The set $S_\theta = \{\operatorname{cis} k\theta \mid k \in \mathbb{N}\}$ is not dense in T if π/θ is rational: if $\pi/\theta = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$, then $\operatorname{cis} k\theta = \operatorname{cis} kq\pi/p$ which can take only $2p$ distinct values for $k = 0, 1, \ldots, 2p - 1$.

So the set $S_\theta = \{\operatorname{cis} k\theta \mid k \in \mathbb{N}\}$ is dense in T if and only if π/θ is irrational.

2.2 Connectedness

2.2.1 Exercise

(a) If A is an interval, here it means that there are two real numbers α and β and one of the following cases holds: $A = (\alpha, \beta)$, $A = (\alpha, \beta]$, $A = [\alpha, \beta)$, $A = [\alpha, \beta]$. In the first case, for instance, if $a, b \in A$ and $a < b$, then $a > \alpha$ and $b < \beta$, so if $a \leq x \leq b$ then also $x > \alpha$ and $x < \beta$, so $x \in A$. The same in the other cases.

(b) Simply apply Theorem 2.3.

2.2.2 Exercise

Let $\phi(s) = sb + (1-s)a$ for $s \in [0,1]$. If $s_0 \in S = \{s \in [0,1] \mid \phi(s) \in A\}$, then $\bar{a} = \phi(s_0) \in A$; as A is an open subset *of* G, there is a ball $B_G(\bar{a}, r) = \{z \in G \mid |z - \bar{a}| < r\}$ such that $B_G(\bar{a}, r) \subseteq A$. So if $|s - s_0| < r/|b-a|$, then $|\phi(s) - \phi(s_0)| < r$, which yields that $\phi(s) \in A$, and we get $B(s_0, r/|b-a|) \subseteq S$.

The same for T.

2.2.3 Exercise

(a) X is connected. Let $A = \{z \in \mathbb{C} \mid |z| < 1\}$ and $B = \{z \in \mathbb{C} \mid |z - 2| < 1\}$; then $X = \overline{A} \cup B$. A is connected and so is $\overline{A}$ by Proposition 2.8 (a); B is connected; $1 \in \overline{A} \cap B$, so by Lemma 2.6 X is connected.

(b) X is not connected. If $x \in [0,1)$, the component of x must be an interval I and $[0,1) \subseteq I$; since also $I \subseteq X$, the only possibility is $I = [0,1)$.

Since for $n \geq 1$

$$1 + \frac{1}{n+1} < 1 + \frac{1}{n} + \frac{1}{2n(n+1)} < 1 + \frac{1}{n} < 1 + \frac{1}{n} + \frac{1}{2n(n-1)} < 1 + \frac{1}{n-1} \tag{2.19}$$

and

$$\frac{1}{2n(n+1)} < \frac{1}{2n(n-1)} \tag{2.20}$$

we have $B(\frac{1}{n}, \frac{1}{2n(n+1)}) \cap X = \left\{1 + \frac{1}{n}\right\}$ so the component of $1 + \frac{1}{n}$ is $\left\{1 + \frac{1}{n}\right\}$.

(c) X is not connected and its components are

$$C_k = \{r \operatorname{cis} \theta \mid 2k\pi < \theta < 2(k+1)\pi, \theta < r < \theta + 2\pi, k \in \mathbb{N}\} \tag{2.21}$$

but to prove it without arcwise connectedness would be a folly.

2.2.4 Exercise

If D were not connected, there would be at least two components C_1 and C_2 of D. If $x_1 \in C_1$, $x_2 \in C_2$, there must be D_{j_1} and D_{j_2} such that $x_1 \in D_{j_1}$ and $x_2 \in D_{j_2}$, and since D_{j_1} and D_{j_2} are connected, we have $D_{j_1} \subseteq D_1$ and $D_{j_2} \subseteq D_2$. But since $D_{j_1} \cap D_{j_2} \neq \emptyset$, also $D_1 \cap D_2 \neq \emptyset$ holds, which is impossible.

2.2.5 Exercise

Take $a \in F$ and for $\epsilon > 0$ call A_ϵ the set of all points b of F such that there are points $z_0, z_1, \ldots, z_n$ in F with $z_0 = a$, $z_n = b$ and $d(z_{k-1}, z_k) < \epsilon$ for $1 \leq k \leq n$. The set A_ϵ is open in F, since if $x \in A_\epsilon$ and $y \in B_F(x, \epsilon)$ clearly $y \in A_\epsilon$. But $F - A_\epsilon$ is open in F too: if $x \in F - A_\epsilon$ and there were $y \in A_\epsilon \cap B_F(x, \epsilon)$ then $x \in A_\epsilon$, a contradiction, so $B_F(x, \epsilon) \subseteq F - A_\epsilon$. Now we have that if $F - A_\epsilon \neq \emptyset$ then F is not connected, since F it is connected, then $F - A_\epsilon = \emptyset$ and $A_\epsilon = F$. Being ϵ any positive real number, the statement is proved. It looks like the fact that F is closed is not needed.

The set $F = \left\{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y = 1/|x|\right\}$ satisfies the given condition and is closed, but it is not connected.

2.3 Sequences and completness

2.3.1 Exercise

(a) If A is closed, by Proposition 3.2 A contains all the points to which some sequence in A converges, in particular all its limit points.

If A is not closed, take $x \in \overline{A} - A$. If $x_0 \in A$, then $d(x, x_0) > 0$, so take $n_1 = \min\{n \in \mathbb{N} \mid 1/n < d(x, x_0)\}$; since $x_0 \notin B(x, 1/n_1)$, there is a point x_1 such that $x_1 \in B_A(x, 1/n_1)$ and $x_1 \neq x_0$. Going on in this way, we have a sequence x_k in A whose points are all distinct, a sequence of natural numbers n_k such that $n_k < n_{k+1}$ for each k, and $\lim x_k = x$: if $\epsilon > 0$, there is $\bar{k}$ such that $1/n_{\bar{k}} < \epsilon$, so $x_{\bar{k}} \in B_A(x, \epsilon)$. This shows that x is a limit point of A, so if A is not closed, it does not contain all its limit points.

(b) Call A^l the set of all limit points of A. From the proof of point (a) we know that if $x \in \overline{A} - A$ then $x \in A^l$, so $\overline{A} \subseteq A \cup A^l$.

Now take $x \in A \cup A^l$. If $x \in A$, then $x \in \overline{A}$. If $x \notin \overline{A}$, there is a ball $B(x, r)$ of X such that $B(x, r) \cap A = \emptyset$, so $x \notin A^l$; then $x \in A^l$ implies $x \in \overline{A}$. Eventually $A \cup A^l \subseteq \overline{A}$.

2.3.2 Exercise

If x_n is a Cauchy sequence in Y, it is a Cauchy sequence in X too, since the metric is the same, so $\lim x_n = x$ for some $x \in X$. By Proposition 3.2 $x \in Y$, and Y is complete.

2.3.3 Exercise

If $x \in \overline{A}$ and $y \in \overline{A}$, for every $\epsilon > 0$ there are two points $\bar{x} \in A$ and $\bar{y} \in A$ such that $d(x, \bar{x}) < \epsilon$ and $d(y, \bar{y}) < \epsilon$. Then for every $\epsilon > 0$ we have $d(x, y) \leq d(x, \bar{x}) + d(\bar{x}, \bar{y}) + d(\bar{y}, y) < d(\bar{x}, \bar{y}) + 2\epsilon \leq \operatorname{diam} A + 2\epsilon$, which implies $d(x, y) \leq \operatorname{diam} A$, whence $\operatorname{diam} \overline{A} \leq \operatorname{diam} A$. Since $A \subseteq \overline{A}$ we have also $\operatorname{diam} A \leq \operatorname{diam} \overline{A}$, then $\operatorname{diam} A = \operatorname{diam} \overline{A}$.

2.3.4 Exercise

We know that

$$d(z_n, z) = \frac{2\,|z_n - z|}{\sqrt{(1 + |z_n|^2)(1 + |z|^2)}}. \tag{2.22}$$

Since $||z_n| - |z|| \leq |z_n - z|$, if $|z_n - z| \to 0$ then $|z_n| \to |z|$, and $d(z_n, z) \to 0$. Since $d(z, \infty) - d(z_n, z) \leq d(z_n, \infty) \leq d(z_n, z) + d(z, \infty)$, if $d(z_n, z) \to 0$ then $d(z_n, \infty) \to d(z, \infty)$, or

$$\frac{2}{\sqrt{1 + |z_n|^2}} \to \frac{2}{\sqrt{1 + |z|^2}} \tag{2.23}$$

which yields again $|z_n| \to |z|$ and $|z_n - z| = d(z_n, z)\sqrt{(1 + |z_n|^2)(1 + |z|^2)} \to 0$.

Now suppose $|z_n| \to +\infty$. Since for $z \in \mathbb{C}$ and $v \in \mathbb{C}$ we have the inequality $|z - v| \leq \sqrt{2}\sqrt{|z|^2 + |v|^2}$, we get for $n \in \mathbb{N}$, $m \in N$ that

$$\begin{aligned}
d(z_n, z_m) &\leq \frac{\sqrt{2}\sqrt{|z_n|^2 + |z_m|^2}}{\sqrt{(1 + |z_n|^2)(1 + |z_m|^2)}} = \\
&= \frac{\sqrt{2}\sqrt{\left(\frac{1}{|z_m|}\right)^2 + \left(\frac{1}{|z_n|}\right)^2}}{\sqrt{\left(\left(\frac{1}{|z_n|}\right)^2 + 1\right)\left(\left(\frac{1}{|z_m|}\right)^2 + 1\right)}} \leq \\
&\leq \sqrt{2}\sqrt{\left(\frac{1}{|z_m|}\right)^2 + \left(\frac{1}{|z_n|}\right)^2}
\end{aligned}$$

Since for any $\epsilon > 0$ there is N such that $k > N$ implies $|z_n| > 2/\epsilon$, for $n > N$ and $m > N$ we have

$$d(z_n, z_m) < \sqrt{2}\sqrt{2(\epsilon/2)^2} = \epsilon. \tag{2.24}$$

Since

$$d(z_n, \infty) = \frac{2}{\sqrt{1 + |z_n|^2}} \tag{2.25}$$

of course if $|z_n| \to +\infty$ then $d(z_n, \infty) \to 0$, or $z_n \to \infty$. So z_n surely is convergent *in* $\mathbb{C}_\infty$.

2.3.5 Exercise

Let x_n be a convergent sequence in any metric space (X, d), and $\lim x_n = x$. This means that for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \epsilon/2$. Then for $n \geq N$ and $m \geq N$ we have $d(x_n, x_m) \leq d(x_n, x) + d(x_m, x) = \epsilon$.

2.3.6 Exercise

Any open proper subset of $\mathbb{R}^n$, for example $B(O, r)$ with $r > 0$, or the half-space $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > 0\}$. Also $(\mathbb{C}, d)$ where d is the metric on $\mathbb{C}$.

2.3.7 Exercise

Take

$$d(x,y)=\frac{2\,|x-y|}{\sqrt{x^2+1}\sqrt{y^2+1}}. \tag{2.26}$$

This amounts to project $X \in S^1-\{(0,1)\}$, where

$$S^1=\left\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2=1\right\}, \tag{2.27}$$

onto the x_1 axis, and take as distance between two points on that axis the Euclidean distance between the corresponding points on S^1. Since $d(x,y)\leq 2\,|x-y|$, it is obvious that $|x-y|\to 0$ implies $d(x,y)\to 0$.

To show the other implication, this time without using the point ∞, we start proving the inequality

$$\frac{|x-y|}{\sqrt{x^2+1}\sqrt{y^2+1}}\geq\left|\frac{1}{\sqrt{x^2+1}}-\frac{1}{\sqrt{y^2+1}}\right| \tag{2.28}$$

that holds for every $x\in\mathbb{R}$ and $y\in\mathbb{R}$; if $x=-y$ the inequality becomes

$$\frac{|x|}{\sqrt{x^2+1}}\geq 0 \tag{2.29}$$

which is true; if $x\neq -y$

$$\begin{aligned}
|x-y| &= \frac{|x-y|\,|x+y|}{|x+y|}=\frac{\left|x^2-y^2\right|}{|x+y|}\geq\frac{\left|x^2-y^2\right|}{|x|+|y|}\geq\\
&\geq\frac{\left|x^2-y^2\right|}{\sqrt{x^2+1}+\sqrt{y^2+1}}=\\
&=\frac{\left|\sqrt{x^2+1}-\sqrt{y^2+1}\right|\left|\sqrt{x^2+1}+\sqrt{y^2+1}\right|}{\left|\sqrt{x^2+1}+\sqrt{y^2+1}\right|}=\\
&=\left|\sqrt{x^2+1}-\sqrt{y^2+1}\right|
\end{aligned}$$

whence

$$\frac{|x-y|}{\sqrt{x^2+1}\sqrt{y^2+1}}\geq\frac{\left|\sqrt{x^2+1}-\sqrt{y^2+1}\right|}{\sqrt{x^2+1}\sqrt{y^2+1}}=\left|\frac{1}{\sqrt{x^2+1}}-\frac{1}{\sqrt{y^2+1}}\right|. \tag{2.30}$$

So if $d(x_n,x)\to 0$ we have

$$\left|\frac{1}{\sqrt{x_n^2+1}}-\frac{1}{\sqrt{x^2+1}}\right|\leq\frac{|x_n-x|}{\sqrt{x_n^2+1}\sqrt{x^2+1}}=\frac{1}{2}d(x_n,x) \tag{2.31}$$

which yields $\sqrt{x_n^2+1}\to\sqrt{x^2+1}$ and

$$|x_n-x|=\frac{1}{2}\sqrt{x_n^2+1}\sqrt{x^2+1}d(x_n,x)\to 0. \tag{2.32}$$

To show that x_n is a Cauchy sequence if $|x_n|\to 0$, we use once again the inequality

$$d(x_n,x_m)\leq\sqrt{2}\sqrt{\left(\frac{1}{|x_m|}\right)^2+\left(\frac{1}{|x_n|}\right)^2}. \tag{2.33}$$

2.3.8 Exercise

Take $\epsilon>0$. Suppose $\lim x_{n_k}=x$, then there is an $N_1\in\mathbb{N}$ such that $k\geq N_1$ implies $d(x_{n_k},x)<\frac{\epsilon}{2}$. Since x_n is a Cauchy sequence, there is an $N_2\in\mathbb{N}$ such that $n\geq N_2$ and $m\geq N_2$ implies $d(x_n,x_m)<\frac{\epsilon}{2}$. Also, there is an $N_3\in\mathbb{N}$ such that $k\geq N_3$ implies $n_k\geq N_2$. If $N=\max\{N_1,N_3\}$, then $k\geq N$ implies

$$d(x_n,x)\leq d(x_n,x_{n_k})+d(x_{n_k},x)<d(x_n,x_{n_k})+\frac{\epsilon}{2} \tag{2.34}$$

for every $n\in\mathbb{N}$, and with $n_k\geq N_2$, so if $n\geq N_2$ we have $d(x_n,x)<\epsilon$.

2.4 Compactness

2.4.1 Exercise

We need to prove that if for any collection $\mathcal{F}$ of closed subset of K with the finite intersection property results

$$\bigcap_{C\in\mathcal{F}} C \neq \emptyset \tag{2.35}$$

then K is compact. Suppose there are open subsets A_i of X such that

$$K \subseteq \bigcup_{i\in I} A_i \tag{2.36}$$

and that for any finite subset J of I results

$$K \nsubseteq \bigcup_{j\in J} A_j; \tag{2.37}$$

then for any finite subset J of I there is $x \in K$ such that

$$x \notin \bigcup_{j\in J} A_j \tag{2.38}$$

which implies that for every $j \in J$ $x \in K - A_j$; the sets $K - A_j = K \cap (X - A_j)$ are closed in K, so what we have just proved is that the collection $K - A_J \mid j \in J$ has the finite intersection property; by hypothesis, there is a y such that

$$y \in \bigcap_{j\in J} K - A_j = K - \bigcup_{j\in J} A_j \tag{2.39}$$

and this is a contradiction.

2.4.2 Exercise

Since $p \in R$ and $q \in R$ it is obvious that $\operatorname{diam} R \geq d(p,q)$. If $x \in R$ and $y \in R$, then for $i = 1,\dots,n$ we have $p_i \leq x_i \leq q_i$ and $p_i \leq y_i \leq q_i$ whence $|y_i - x_i| \leq q_i - p_i$ for $i = 1,\dots,n$ and

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} \leq \sqrt{\sum_{i=1}^{=}(q_i - p_i)^2 d(p,q)} \tag{2.40}$$

so $\operatorname{diam} R \leq d(p,q)$.

2.4.3 Exercise

Choose $m \in \mathbb{N}$ such that $d(a,b)/m < \epsilon$ and take the points

$$x^{(k_1,\dots,k_n)} = \left(a_1 + \frac{b_1 - a_1}{m}k_1, \dots, a_n + \frac{b_1 - a_1}{m}k_n\right),\ 0 \leq k_1,\dots,k_n \leq m. \tag{2.41}$$

Define the m^n rectangles

$$R^{(k_1,\dots,k_n)} = \times_{i=1}^{n}[x_i^{(k_1,\dots,k_n)}, x_i^{(k_1+1,\dots,k_n+1)}],\ 0 \leq k_1,\dots,k_n \leq m-1. \tag{2.42}$$

These rectangles are all subsets of F, and

$$\begin{aligned}\operatorname{diam} R^{(k_1,\dots,k_n)} &= d(x_i^{(k_1,\dots,k_n)}, x_i^{(k_1+1,\dots,k_n+1)}) = \\ &= \sqrt{\sum_{i=1}^{n}\left(x_i^{(k_1+1,\dots,k_n+1)} - x_i^{(k_1,\dots,k_n)}\right)^2} = \\ &= \sqrt{\sum_{i=1}^{n}\frac{(b_i - a_i)^2}{m^2}} = \frac{d(a,b)}{m} < \epsilon.\end{aligned}$$

Furthermore,

$$R = \bigcup_{(k_1,\ldots,k_n)=(0,\ldots,0)}^{(m-1,\ldots,m-1)} R^{(k_1,\ldots,k_n)}. \tag{2.43}$$

In fact, if $x \in R^{(k_1,\ldots,k_n)}$ then, since $0 \le k_i \le m-1$, for $1 \le i \le n$ we have

$$a_i \le a_i + \frac{b_i - a_i}{m} k_i \le x_i \le a_i + \frac{b_i - a_i}{m}(k_i + 1) \le b_i; \tag{2.44}$$

if $x \in R$, then $a_i \le x_i \le b_i$ for $0 \le i \le n$, so there surely are k_i for $0 \le i \le n$ such that

$$a_i + \frac{b_i - a_i}{m} k_i \le x_i \le a_i + \frac{b_i - a_i}{m}(k_i + 1). \tag{2.45}$$

Eventually, if A is a set, $x \in A$ and $\operatorname{diam} A = r$ then $A \subseteq B(x, r)$, since for every $y \in A$ we have $d(x, y) \le r$.

2.4.4 Exercise

If F is union of a finite number of compact set, say

$$F = \bigcup_{i=1}^{n} F_i \tag{2.46}$$

and $A_j,\ j \in J$ is a collection of open sets such that

$$F \subseteq \bigcup_{j \in J} A_j \tag{2.47}$$

then also

$$F_i \subseteq \bigcup_{j \in J} A_j \quad i = 1, \ldots, n \tag{2.48}$$

holds, so for each $i = \ldots, n$ there is a finite subset $\bar{J}_i$ of J such that

$$F_i \subseteq \bigcup_{j \in \bar{J}_i} A_j \quad i = 1, \ldots, n; \tag{2.49}$$

the set $\bar{J} = \cup_{j=1}^{n} \bar{J}_i$ is also finite, and

$$F \subseteq \bigcup_{j \in \bar{J}} A_j. \tag{2.50}$$

2.4.5 Exercise

First, we observe that for every $x, y, h \in X$ we have $d(x, y) = d(x + h, y + h)$, since

$$\begin{aligned} d(x, y) &= \sup_{n \in \mathbb{N}} |y_n - x_n| = \sup_{n \in \mathbb{N}} |(y_n + h_n) - (x_n + h_n)| = \\ &= \sup_{n \in \mathbb{N}} |(y + h)_n - (x + h)_n| = d(x + h, y + h). \end{aligned}$$

Then, that

$$\bar{B}(x, \epsilon) \subseteq \bigcup_{k=1}^{n} B(x_k, \delta) \tag{2.51}$$

if and only if

$$\bar{B}(0, \epsilon) \subseteq \bigcup_{k=1}^{n} B(x_k - x, \delta). \tag{2.52}$$

Indeed suppose the latter inclusion holds. Then $y \in \bar{B}(x,\epsilon) \Rightarrow d(x,y) \le \epsilon \Rightarrow d(0, y-x) \le \epsilon \Rightarrow y - x \in \bar{B}(0,\epsilon)$ so

$$y - x \in \bigcup_{k=1}^{n} B(x_k - x, \delta) \tag{2.53}$$

and there is k such that $d(x_k - x, y - x) = d(x_k, y) < \delta$ whence $y \in B(x_k, \delta)$ and

$$y \in \bigcup_{k=1}^{n} B(x_k, \delta) \tag{2.54}$$

and the former inclusion also holds. The same to prove that the former inclusion implies the latter one.

So we have proved that for every $x \in X$ $\bar{B}(x,\epsilon)$ is totally bounded if and only if $\bar{B}(0,\epsilon)$ is totally bounded.

Now we prove that X is complete. Take $x^{(k)}$ in X: that is, $x^{(k)}$ is a sequence, whose elements are sequences of complex numbers; suppose $x^{(k)}$ is Cauchy, that is, for every $\delta > 0$ there is $N \in \mathbb{N}$ such that $h, l \ge N$ implies $d(x^{(h)}, x^{(l)}) = \sup_{n\in\mathbb{N}} \left| x_n^{(h)} - x_n^{(l)} \right| < \delta$. This means that for every $n \in \mathbb{N}$ the sequence in $\mathbb{C}$: $k \mapsto x_n^{(k)}$ is Cauchy, and, since $\mathbb{C}$ is complete, there is a complex number x_n such that

$$\lim_{k\to+\infty} x_n^{(k)} = x_n. \tag{2.55}$$

Now we show that $x^{(k)}$ converges to x in X: again since $x^{(k)}$ is Cauchy, for every $\delta > 0$ there is $N \in \mathbb{N}$ such that $h, l \ge N$ implies $d(x^{(h)}, x^{(l)}) < \delta$, in particular, for every $h \ge N$ and for every $n \in \mathbb{N}$

$$\lim_{l\to+\infty} \left| x_n^{(h)} - x_n^{(l)} \right| = \left| x_n^{(h)} - x_n \right| < \delta \tag{2.56}$$

which shows that for every $h \ge N$

$$d(x_n^{(h)}, x_n) = \sup_{n\in\mathbb{N}} \left| x_n^{(h)} - x_n \right| < \delta. \tag{2.57}$$

Since we proved that X is complete and $\bar{B}(0,\epsilon)$ is closed in X, $\bar{B}(0,\epsilon)$ is also complete.

Now define the sequence $x^{(k)}$ as:

$$x_n^{(k)} = \begin{cases} \epsilon & \text{if } n = k \\ 0 & \text{if } n \ne k \end{cases}. \tag{2.58}$$

Clearly for every $k \in \mathbb{N}$ $x^{(k)} \in \bar{B}(0,\epsilon)$ and if $h \ne k$ then $d(x^{(k)}, x^{(h)}) = \epsilon$, so for any point $x \in \bar{B}(0,\epsilon)$ the ball $B(x, \epsilon/2)$ contains at most only one point $x^{(k)}$, and there is no finite number of such balls that contains all $\bar{B}(0,\epsilon)$.

2.4.6 Exercise

Suppose A is a totally bounded subset of a metric space (X, d) and choose $\epsilon > 0$. There are points $x_k,\ k = 1, \ldots, n$ in A such that

$$A \subseteq \bigcup_{k=1}^{n} B\left(x_k, \frac{\epsilon}{2}\right) \tag{2.59}$$

so

$$\overline{A} \subseteq \bigcup_{k=1}^{n} \bar{B}\left(x_k, \frac{\epsilon}{2}\right) \subseteq \bigcup_{k=1}^{n} B\left(x_k, \epsilon\right). \tag{2.60}$$

2.5 Continuity

2.5.1 Exercise

$(a) \Rightarrow (b)$ Take $\epsilon > 0$. Since f is continuous at a, there is δ such that $0 < d(a,x) < \delta \Rightarrow \rho(\alpha, f(x)) < \epsilon$. So $x \in B_X(a,\delta) \Rightarrow f(x) \in B_\Omega(\alpha,\epsilon) \Rightarrow x \in f^{-1}(B_\Omega(\alpha,\epsilon))$.

$(b) \Rightarrow (c)$ Take x_n in X such that $\lim x_n = a$ and $\epsilon > 0$. By hypothesis there is a ball $B_X(a, \delta)$ such that $B_X(a, \delta) \subseteq f^{-1}(B_\Omega(\alpha, \epsilon))$ and there is $N \in \mathbb{N}$ such that $n \geq N \Rightarrow d(a, x_n) < \delta \Rightarrow x_n \in B_X(a, \delta)$ so $n \geq N \Rightarrow f(x_n) \in B_\Omega(\alpha, \epsilon) \Rightarrow \rho(\alpha, x_n) < \epsilon$.

$(c) \Rightarrow (a)$ Suppose f is not continuous at a, that is, there is $\epsilon > 0$ such that for every $\delta > 0$ there is $x \in X$ such that $d(a, x) < \delta$ and $\rho(\alpha, f(x)) \geq \epsilon$. In particular for every $n \in \mathbb{N}$ we can choose $x_n \in X$ such that $d(a, x_n) < 1/n$ and $\rho(\alpha, f(x_n)) \geq \epsilon$. Clearly $\lim x_n = a$ but $f(x_n)$ either has no limit or its limit is not α.

2.5.2 Exercise

If $f, g : X \to \mathbb{C}$ are uniformly continuous maps for every $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for each $x, y \in X$ $d(x, y) < \delta_1 \Rightarrow |f(x) - f(y)| < \epsilon/2$ and $d(x, y) < \delta_2 \Rightarrow |g(x) - g(y)| < \epsilon/2$. Then for each $x, y \in X$ $d(x, y) < \min\{\delta_1, \delta_2\} \Rightarrow |(f + g)(x) - (f + g)(y)| = |(f(x) - f(y)) + (g(x) - g(y))| \leq |f(x) - f(y)| + |g(x) - g(y)| < \epsilon$.

If $f, g : X \to \mathbb{C}$ are Lipschitz maps there are $M_1 > 0, M_2 > 0$ such that for all $x, y \in X$ $|f(x) - f(y)| \leq M_1 d(x, y)$ and $|g(x) - g(y)| \leq M_2 d(x, y)$, so $|(f + g)(x) - (f + g)(y)| = |(f(x) - f(y)) + (g(x) - g(y))| \leq M_1 d(x, y) + M_2 d(x, y) = (M_1 + M_2) d(x, y)$.

2.5.3 Exercise

If $f, g : X \to \mathbb{C}$ are bounded uniformly continuous maps then there are $M_1 > 0$ and $M_2 > 0$ such that for every $x \in X$ we have $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$; furthermore for every $\epsilon > 0$ there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for each $x, y \in X$ $d(x, y) < \delta_1 \Rightarrow |f(x) - f(y)| < \epsilon/(M_1 + M_2)$ and $d(x, y) < \delta_2 \Rightarrow |g(x) - g(y)| < \epsilon/(M_1 + M_2)$.

Now if $d(x, y) < \min\{\delta_1, \delta_2\}$

$$\begin{aligned}
|(fg)(x) - (fg)(y)| &= |[f(x) - f(y)]g(x) + [g(x) - g(y)]f(y)| \leq \\
&\leq |f(x) - f(y)|\,|g(x)| + |g(x) - g(y)|\,|f(y)| \leq \\
&\leq M_1 \frac{\epsilon}{M_1 + M_2} + M_2 \frac{\epsilon}{M_1 + M_2} = \epsilon.
\end{aligned}$$

If $f, g : X \to \mathbb{C}$ are bounded Lipschitz maps there are $M_1 > 0$ and $M_2 > 0$ such that for every $x \in X$ we have $|f(x)| \leq M_1$ and $|g(x)| \leq M_2$; furthermore there are $N_1 > 0, N_2 > 0$ such that for all $x, y \in X$ $|f(x) - f(y)| \leq N_1 d(x, y)$ and $|g(x) - g(y)| \leq N_2 d(x, y)$.

So

$$\begin{aligned}
|(fg)(x) - (fg)(y)| &= |[f(x) - f(y)]g(x) + [g(x) - g(y)]f(y)| \leq \\
&\leq N_1 d(x, y) M_1 + N_2 d(x, y) M_2 = \\
&= (N_1 M_1 + N_2 M_2) d(x, y).
\end{aligned}$$

2.5.4 Exercise

If $g : X \to Y$ and $f : Y \to Z$ are uniformly continuous maps, for every $\epsilon > 0$ there is $\delta > 0$ such that $d_Y(x, y) < \delta \Rightarrow d_Z(f(x), f(y)) < \epsilon$, and there is $\theta > 0$ such that $d_X(u, v) < \theta \Rightarrow d_Y(g(u), g(v)) < \delta$, so $d_X(u, v) < \theta \Rightarrow d_Z(f(g(u)) - f(g(v))) = d_Z((f \circ g)(u) - (f \circ g)(v)) < \epsilon$.

If $f : X \to Y$ and $g : Y \to Z$ are Lipschitz maps, there are M and N such that $d_Z(f(x), f(y)) \leq N d_Y(x, y)$ for every $x, y \in Y$ and $d_Y(g(u), g(v)) \leq M d_X(u, v)$ for every $u, v \in X$, so

$$\begin{aligned}
d_Z((f \circ g)(u), (f \circ g)(v)) &= d_Z(f(g(u)), f(g(v))) \leq \\
&\leq N d_Y(g(u), g(v)) \leq N M d_X(u, v).
\end{aligned}$$

2.5.5 Exercise

Take $\epsilon > 0$. Since f is uniformly continuous, there is δ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$; since x_n is a Cauchy sequence there is $\bar{n} \in \mathbb{N}$ such that $n \geq \bar{n}, m \geq \bar{n} \Rightarrow d(x_n, x_m) < \delta \Rightarrow d(f(x_n), f(x_m)) < \epsilon$.

No. Take $X = \mathbb{R} - \{0\}$, $\Omega = \mathbb{R}$, $x_n = 1/n$, $f(x) = 1/x$. Clearly x_n is a Cauchy sequence, since it is convergent in $\mathbb{R}$, and f is continuous, but $f(x_n) = n$ is not a Cauchy sequence, since it is not convergent.

2.5.6 Exercise

Since D is dense in X, for every $x \in X$ there is a sequence x_n *in* D such that $\lim x_n = x$. Exercise 5 allows to say that $f(x_n)$ is a Cauchy sequence in Ω, and since Ω is complete, $f(x_n)$ is convergent. Let's show that $\lim x_n = \lim y_n \Rightarrow \lim f(x_n) = \lim f(y_n)$. Say $l = \lim x_n = \lim y_n$, and take $\epsilon > 0$. Since f is uniformly continuous, there is δ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$, and there are $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow d(l, x_n) < \delta/2$ and $m \geq N_2 \Rightarrow d(l, y_m) < \delta/2$. So $k \geq \max\{N_1, N_2\} \Rightarrow d(x_k, y_k) \leq d(l, x_k) + d(l, y_k) = \delta$ and $k \geq \max\{N_1, N_2\} \Rightarrow d(f(x_k), f(y_k)) < \epsilon$, which yields also $d(\lim x_k, \lim y_k) < \epsilon$. Since ϵ is any positove real number, this proves that the two limits are equal.

Now we can define $g : X \to \Omega$ as $g(x) = \lim f(x_k)$ where x_k is any sequence in D such that $\lim x_k = x$. To see that $g_{|D} = f$ is easy: if $x \in D$ take $x_k = x$ for all $k \in N$. We have still to show that g is uniformly continuous. Take $\epsilon > 0$; since f is uniformly continuos there is δ such that $d(u, v) < \delta \Rightarrow d(f(u), f(v)) < \epsilon$ for any two points $u, v \in D$; take $x, y \in X$ such that $d(x, y) < \delta/3$, and let x_k, y_k be sequences in D such that $\lim x_k = x$, $\lim y_k = y$, so there are $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that $n \geq N_1 \Rightarrow d(x, x_n) < \delta/3$ and $m \geq N_2 \Rightarrow d(y, y_m) < \delta/3$; then if $N = \max\{N_1, N_2\}$ we have $k \geq N \Rightarrow d(x_k, y_k) \leq d(x, x_k) + d(x, y) + d(y, y_k) < \delta$. So for $k \geq N$ we have also $d(g(x), g(y)) \leq d(g(x), f(x_k)) + d(f(x_k), f(y_k)) + d(g(y), f(y_k)) < d(g(x), f(x_k)) + \epsilon + d(g(y), f(y_k))$. Since this inequality holds for every $k \geq N$, eventually we get, for every $x, y \in X$ such that $d(x, y) < \delta/3$, $d(g(x), g(y)) < \lim[d(g(x), f(x_k)) + \epsilon + d(g(y), f(y_k))] = \epsilon$.

2.5.7 Exercise

Of course it is enough to prove the statement in the case that the polygon P is made of only one segment. Let $L = d(P, \mathbb{C} - G)$. Since $\mathbb{C} - G$ is closed, P is compact by Theorem 4.10, and $\mathbb{C} - G$ and P are disjoint, then $L > 0$. Take $n \in \mathbb{N}$ such that $r = d(a, b)/n < L/2$, put $x_0 = a, x_n = b$ and choose $n - 1$ points x_i $1 \leq i \leq n - 1$ on P such that $d(x_i, x_{i+1}) = r, 0 \leq i \leq n - 1$. We have $B(x_i, L) \subseteq G, 0 \leq i \leq n - 1$ and since $r < L$ also $x_{i+1} \in B(x_i, L), 0 \leq i \leq n - 1$. Now it is obvious that x_i and x_{i+1} can be joined by a polygon which is composed of one line segment parallel to the real axis and of one line segment parallel to the imaginary axis, so a and b can be joined by a polygon which is composed of n line segments parallel to the real axis and of n line segments parallel to the imaginary axis.

N.B. What has Theorem 5.15 got to do with this proof?

2.5.8 Exercise

For every $\epsilon > 0$ we have

$$X \subseteq \bigcup_{\omega \in \Omega} f^{-1}\left(B_\Omega\left(\omega, \frac{\epsilon}{2}\right)\right); \tag{2.61}$$

since this is an open cover of X, which is compact and so sequentially compact by Theorem 4.9, Lebesgue's Covering Lemma says that there is a δ such that for every $x \in X$ we have $B_X(x, \delta) \subseteq f^{-1}(B_\Omega(\omega, \epsilon/2))$ for some $\omega \in \Omega$. Now if $x, y \in X$ are such that $d_X(x, y) < \delta$, surely $y \in B_X(x, \delta)$, so both $x \in f^{-1}(B_\Omega(\omega, \epsilon/2))$ and $y \in f^{-1}(B_\Omega(\omega, \epsilon/2))$ hold for some $\omega \in \Omega$, which implies both $f(x) \in B_\Omega(\omega, \epsilon/2)$ and $f(y) \in B_\Omega(\omega, \epsilon/2)$, that is, $d_\Omega(f(x), f(y)) < \epsilon$.

2.5.9 Exercise

If X is disconnected, then $X = Y \cup Z$ where Y and Z are open, disjoint and not empty; clearly, they are also closed. By Proposition 4.3 Y is compact (also Z, but we don't need it). There are two points $y \in Y$ and $z \in Z$. Now take points $x_0, \ldots, x_n$ with $x_0 = y$ and $x_n = z$. Surely there is $\bar{k}$ such that $x_{\bar{k}}$ is the last among these points which belongs to Y, that is

$$\bar{k} = \max\{k \in \mathbb{N} \mid x_k \in Y\} \tag{2.62}$$

so $x_k \in Z$ if $k > \bar{k}$. By Theorem 5.17 $r = d(Y, Z) > 0$, then $d(x_{\bar{k}}, x_{\bar{k}+1}) \geq r$. Conclusion: no matter how the points x_k are chosen, there are always two of them whose sisctance is not less than r, contradicting the hypothesis.

2.5.10 Exercise

If $x \in X$ there is a sequence x_n in D such that $\lim x_n = x$. So $\lim f(x_n) = f(x)$ and $\lim g(x_n) = g(x)$, since f and g are continuous. But for every $n \in \mathbb{N}$ we have $f(x_n) = g(x_n)$, since $x_n \in D$, so $f(x) = g(x)$.

That the function whose existence is proved in Exercise 6 is unique is an immediate consequence.

2.6 Uniform convergence

2.6.1 Exercise

Exactly as in the proof of Theorem 6.1. Take $\epsilon > 0$, since $f = \lim f_n$, there is n such that $\rho(f(x), f_n(x)) < \epsilon/3$ for all $x \in X$; since f_n is uniformly continuous, there is δ such that $d(x, y) < \delta$ implies $\rho(f_n(x), f_n(y)) < \epsilon/3$, so if $d(x, y) < \delta$ we have $\rho(f(x), f(y)) \leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f(y), f_n(y)) < \epsilon$.

If there are M_n such that $\rho(f_n(x), f_n(y) \leq M_n d(x, y)$ for every $x, y \in X$, $\sup_{n\in\mathbb{N}} M_n = M$ and $f = \lim f_n$, then for every $x, y \in X$ and for every $n \in \mathbb{N}$

$$\begin{aligned}\rho(f(x), f(y)) &\leq \rho(f(x), f_n(x)) + \rho(f_n(x), f_n(y)) + \rho(f(y), f_n(y)) \leq \\ &\leq \rho(f(x), f_n(x)) + \rho(f(y), f_n(y)) + Md(x, y)\end{aligned}$$

so also

$$\rho(f(x), f(y)) \leq \lim[\rho(f(x), f_n(x)) + \rho(f(y), f_n(y)) + Md(x, y)] = Md(x, y). \tag{2.63}$$

Take $X = [0, 1]$, $\Omega = \mathbb{R}$, both with the Euclidean distance, and $f_n(x) = \sqrt{x + 1/n}$. Each f_n is Lipschitz, since for $x \in [0, 1]$

$$|f_n(x) - f_n(y)| \leq \frac{\sqrt{n}}{2} |x - y| \tag{2.64}$$

(note that $\sup M_n = +\infty$), $\lim f_n(x) = \sqrt{x}$ and the convergence is uniform since for $x \in [0, 1]$

$$\left| \sqrt{x + \frac{1}{n}} - \sqrt{x} \right| \leq \sqrt{\frac{1}{n}} \tag{2.65}$$

but the limit f is not Lipschitz; for $f(x) - f(0) = \sqrt{x}$, and if for an M we had $\sqrt{x} \leq Mx$ then $1/\sqrt{x} \leq M$ for every $x \in [0, 1]$, which clearly is false.

Chapter 3

Elementary Properties and Examples of Analytic Functions

3.1 Power series

3.1.1 Exercise

Using the Identity A.1 we have

$$\sum_{n=0}^{k}|c_n| = \sum_{n=0}^{k}\left|\sum_{h=0}^{n} a_h b_{n-h}\right| \le \sum_{n=0}^{k}\sum_{h=0}^{n}|a_h|\,|b_{n-h}| = \sum_{n=0}^{k}\sum_{h=0}^{k-n}|a_n|\,|b_h| \le$$
$$\le \sum_{n=0}^{k}\sum_{h=0}^{k}|a_n|\,|b_h| = \sum_{n=0}^{k}|a_n|\sum_{h=0}^{k}|b_h|$$

and this shows that $\sum c_n$ is absolutely convergent.

Now, let $a_n = x_n + iy_n$ and $b_n = u_n + iv_n$. Then the series $\sum x_n$, $\sum y_n$, $\sum u_n$, $\sum v_n$ are all absolutely convergent, since for any complex number z we have $\mathfrak{Re}\, z \le |z|$ and $\mathfrak{Im}\, z \le |z|$. Let

$$\sum_{k=0}^{+\infty} x_k = X, \quad \sum_{k=0}^{+\infty} y_k = Y, \quad \sum_{k=0}^{+\infty} u_k = U, \quad \sum_{k=0}^{+\infty} v_k = V. \tag{3.1}$$

Now

$$c_k = \sum_{h=0}^{k} a_h b_{k-h} = \sum_{h=0}^{k}(x_h + iy_h)(u_{k-h+iv_{k-h}}) =$$
$$= \left(\sum_{h=0}^{k} x_h u_{k-h} - \sum_{h=0}^{k} y_h v_{k-h}\right) + i\left(\sum_{h=0}^{k} y_h u_{k-h} + \sum_{h=0}^{k} x_h v_{k-h}\right)$$

and

$$\sum_{k=0}^{n} c_k = \left(\sum_{k=0}^{n}\sum_{h=0}^{k} x_h u_{k-h} - \sum_{k=0}^{n}\sum_{h=0}^{k} y_h v_{k-h}\right) +$$
$$+ i\left(\sum_{k=0}^{n}\sum_{h=0}^{k} y_h u_{k-h} + \sum_{k=0}^{n}\sum_{h=0}^{k} x_h v_{k-h}\right).$$

As it is known from the real case, the four series above converge to the corrisponding products, that is

$$\sum_{k=0}^{+\infty} c_k = (XU - YV) + i(YU - XV) = (X + iY)(U + iV) =$$
$$= \left(\sum_{k=0}^{+\infty} a_k\right)\left(\sum_{k=0}^{+\infty} b_k\right).$$

3.1.2 Exercise

What is left to prove follows from Proposition 1.5 and the corresponding statement for the sum of two absolutely convergent series.

3.1.3 Exercise

Straightforwardly:

$$\begin{aligned}\limsup(a_n + b_n) &= \lim \sup_{n\geq k}\{a_n + b_n\} \leq \lim\left(\sup_{n\geq k}\{a_n\} + \sup_{n\geq k}\{b_n\}\right) = \\ &= \limsup a_n + \limsup b_n.\end{aligned}$$

The same for $\liminf$.

3.1.4 Exercise

Even straightforwardlier: for every $k \in \mathbb{N}$ it is obvious that

$$\inf_{n\geq k}\{a_n\} \leq \sup_{n\geq k}\{a_n\} \tag{3.2}$$

whence

$$\lim \inf_{n\geq k}\{a_n\} \leq \lim \sup_{n\geq k}\{a_n\}. \tag{3.3}$$

3.1.5 Exercise

Let's show that $\liminf a_n \geq a$. If $l_i = \liminf a_n < a$, then there would be N such that

$$\inf_{n\geq N}\{a_n\} < \frac{a + l_i}{2} \tag{3.4}$$

which implies that for every $k \geq N$ there is $\bar{k} \geq k$ such that

$$a_{\bar{k}} < \frac{a + l_i}{2} < a \tag{3.5}$$

so it couldn't be $a = \lim a_n$.

Analogously $\limsup a_n \leq a$, so $\liminf a_n \geq \limsup a_n$. Since the opposite inequality always holds, this proves that $\liminf a_n = \limsup a_n$.

3.1.6 Exercise

(a) $\lim \sqrt[n]{|a^n|} = |a|$, so $R = \frac{1}{|a|}$.

(b) $\lim \sqrt[n]{\left|a^{n^2}\right|} = \lim |a|^n$ so

- $R = +\infty$ if $|a| < 1$
- $R = 0$ if $|a| > 1$
- $R = 1$ if $|a| = 1$

(c) $\lim \sqrt[n]{|k^n|} = |k|$ so $R = \frac{1}{|k|}$.

(d) The coefficients are: $a_k = \begin{cases} 1 & \text{if } k = n! \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$. So $\sqrt[k]{|a_k|}$ has a subsequence which takes always the value 1 and a subsequence which takes always the value 0 . Hence $\limsup \sqrt[k]{|a_k|} = 1$ and $R = 1$.

3.1.7 Exercise

The coefficients of this series are:

$$a_k = \begin{cases} \frac{(-1)^n}{n} & \text{if } k = n(n+1) \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

and accordingly

$$\sqrt[k]{|a_k|} = \begin{cases} \frac{1}{n^{\frac{1}{n(n+1)}}} & \text{if } k = n(n+1) \text{ for some } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} . \tag{3.7}$$

Then for any $h \in \mathbb{N}$, $h \geq 2$

$$\sup_{k \geq h} \left\{ \sqrt[k]{|a_k|} \right\} = \frac{1}{n^{\frac{1}{n(n+1)}}} \quad \text{if } n(n+1) \leq h < (n+1)(n+2) \tag{3.8}$$

since for $n \geq 2$ the sequence $n^{\frac{1}{n(n+1)}}$ is decreasing, as can be easily seen deriving the function $x^{\frac{1}{x(x+1)}}$. So $\limsup \sqrt[k]{|a_k|} = 1$.

If $z = 1$, or $z = -1$, being $n(n+1)$ always even, the series becomes

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n} \tag{3.9}$$

which is convergent by Leibnitz criterion.

If $z = i$, the series becomes

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{+\infty} (-1)^n \frac{4n+1}{2n(2n+1)} \tag{3.10}$$

which again is convergent by Leibnitz criterion.

3.2 Analytic functions

3.2.1 Exercise

Writing explicitly the real and immaginary parts of the function:

$$f(x + iy) = u(x, y) + iv(x, y) \tag{3.11}$$

where

$$u(x, y) = x^2 + y^2$$
$$v(x, y) = 0$$

we get

$$u_x(x, y) = 2x$$
$$u_y(x, y) = 2y$$

and

$$v_x(x, y) = 0$$
$$v_y(x, y) = 0$$

so the Cauchy-Riemann equations can be satisfied only for $(x, y) = (0, 0)$.

3.2.2 Exercise

Suppose $a > 0$ and $b > 0$. Since $\lim b_n = b$ and $\limsup a_n = a$, then $\forall \eta > 0 \exists n_1 : n \geq n_1 \Rightarrow b_n < b + \eta$ and $\forall \eta > 0 \exists n_2 : n \geq n_2 \Rightarrow a_n < a + \eta$.

Now

$$a_n b_n = b_n(a_n - a) + a(b_n - b) + ab. \tag{3.12}$$

Since $b > 0$, there is an $n_3 \in \mathbb{N}$ such that $n \geq n_3 \Rightarrow b_n > 0$, so for $n \geq \max\{n_1, n_2, n_3\}$ we have

$$a_n b_n \leq b_n \eta + a\eta + ab. \tag{3.13}$$

Furthermore, there is $M > 0$ such that $b_n \leq M$ for every n, so

$$a_n b_n \leq (M + a)\eta + ab. \tag{3.14}$$

If $\epsilon > 0$, take

$$\eta = \frac{\epsilon}{M + a} \tag{3.15}$$

and for $n \geq \max\{n_1, n_2, n_3\}$ we have

$$a_n b_n \leq ab + \epsilon. \tag{3.16}$$

Since $\lim b_n = b$ and $\limsup a_n = a$, then $\forall \theta > 0 \exists n_1 : n \geq n_1 \Rightarrow b_n > b - \theta$ and $\forall \theta > 0, \forall \bar{n} \in \mathbb{N} \exists n_2 : n_2 > \bar{n} \wedge a_{n_2} > a - \theta$, so for $\bar{n} \geq \max\{n_1, n_3\}$ there is $n_2 > \bar{n}$ such that

$$a_{n_2} b_{n_2} \geq -b_n \eta - a\eta + ab. \tag{3.17}$$

Furthermore, there is n_4 such that $n \geq n_4 \Rightarrow b_n > b/2$; so now we have that for $\bar{n} \geq \max\{n_1, n_3, n_4\}$ there is $n_2 > \bar{n}$ such that

$$a_{n_2} b_{n_2} \geq -\left(\frac{b}{2} + a\right)\eta + ab. \tag{3.18}$$

If $\epsilon > 0$, take

$$\eta = \frac{\epsilon}{b/2 + a} \tag{3.19}$$

and for $\bar{n} \geq \max\{n_1, n_3, n_4\}$ there is $n_2 > \bar{n}$ such that

$$a_{n_2} b_{n_2} \geq ab - \epsilon. \tag{3.20}$$

Of course in the same way it can be proved that, under the same hypotheses, also $\liminf(a_n b_n) = \lim b_n \liminf a_n$ holds.

Now suppose $a < 0$ and $b > 0$. Then

$$-a = \liminf(-a_n) > 0 \tag{3.21}$$

and for what we just proved,

$$\limsup(a_n b_n) = -\liminf(-a_n b_n) = ab. \tag{3.22}$$

If $b < 0$, then $-b = \lim(-b_n) > 0$ and

$$\begin{aligned}\limsup(a_n b_n) &= -\liminf(-a_n b_n) = -\liminf(a_n(-b_n)) = \\ &= -\liminf a_n \lim(-b_n) = -b \liminf a_n\end{aligned}$$

supposing both $\liminf a_n$ and $\limsup a_n$ are finite.

3.2.3 Exercise

What's that supposed to mean? That is, starting from where? After all, it's known from real analysis, isn't it?

3.2.4 Exercise

Since

$$\cos z=\sum_{n=0}^{+\infty}(-1)^n\frac{z^{2n}}{(2n)!} \tag{3.23}$$

then, by Proposition 2.5

$$(\cos z)'=\sum_{n=1}^{+\infty}(-1)^n 2n\frac{z^{2n-1}}{(2n)!}=\sum_{n=1}^{+\infty}(-1)^n\frac{z^{2n-1}}{(2n-1)!}=-\sin z. \tag{3.24}$$

Since

$$\sin z=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{z^{2n-1}}{(2n-1)!} \tag{3.25}$$

then, by Proposition 2.5

$$\begin{aligned}(\sin z)'&=\sum_{n=1}^{+\infty}(-1)^{n+1}(2n-1)\frac{z^{2n-2}}{(2n-1)!}=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{z^{2n-2}}{(2n-2)!}=\\&=\sum_{n=0}^{+\infty}(-1)^n\frac{z^{2n}}{(2n)!}.\end{aligned}$$

3.2.5 Exercise

We have

$$\begin{aligned}e^{iz}&=\sum_{n=0}^{+\infty}\frac{(iz)^n}{n!}=\sum_{n=0}^{+\infty}i^n\frac{z^n}{n!}=\sum_{k=0}^{+\infty}i^{2k}\frac{z^{2k}}{2k!}+\sum_{k=0}^{+\infty}i^{2k+1}\frac{z^{2k+1}}{(2k+1)!}=\\&=\sum_{k=0}^{+\infty}(-1)^k\frac{z^{2k}}{2k!}+i\sum_{k=0}^{+\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}=\cos z+i\sin z,\end{aligned}$$

$$\begin{aligned}e^{-iz}&=\sum_{n=0}^{+\infty}\frac{(-iz)^n}{n!}=\sum_{n=0}^{+\infty}(-i)^n\frac{z^n}{n!}=\\&=\sum_{k=0}^{+\infty}(-i)^{2k}\frac{z^{2k}}{2k!}+\sum_{k=0}^{+\infty}(-i)^{2k+1}\frac{z^{2k+1}}{(2k+1)!}=\\&=\sum_{k=0}^{+\infty}(-1)^k\frac{z^{2k}}{2k!}-i\sum_{k=0}^{+\infty}(-1)^k\frac{z^{2k+1}}{(2k+1)!}=\cos z-i\sin z.\end{aligned}$$

3.2.6 Exercise

1. If $e^{x+iy}=e^x(\cos y+i\sin y)=i$, then it must be $e^x\cos y=0$, $e^x\sin y=1$, and $e^x=|i|=1$, so $x=0$; then $\cos y=0$ and $\sin y=1$, so $y=\frac{\pi}{2}+2k\pi$. Hence
$$\{z\in\mathbb{C}\,|\,e^z=i\}=\left\{i\left(\frac{\pi}{2}+2k\pi\right)\,|\,k\in\mathbb{Z}\right\}. \tag{3.26}$$

2. If $e^{x+iy}=e^x(\cos y+i\sin y)=-1$ we have again $x=0$, and $\cos y=-1$, $\sin y=0$, hence
$$\{z\in\mathbb{C}\,|\,e^z=-1\}=\{i\,(\pi+2k\pi)\,|\,k\in\mathbb{Z}\}. \tag{3.27}$$

3. Now
$$\{z\in\mathbb{C}\,|\,e^z=-i\}=\left\{i\left(\frac{3\pi}{2}+2k\pi\right)\,|\,k\in\mathbb{Z}\right\}. \tag{3.28}$$

4. Finally
$$\{z\in\mathbb{C}\,|\,e^z=0\}=\{i\,(2k\pi)\,|\,k\in\mathbb{Z}\}. \tag{3.29}$$

3.2.7 Exercise

On the one hand

$$\cos(z+w) = \frac{e^{i(z+w)} + e^{-i(z+w)}}{2} = \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw}}{2} \tag{3.30}$$

on the other hand

$$\begin{aligned}\cos z \cos w &= \frac{e^{iz} + e^{-iz}}{2}\frac{e^{iw} + e^{-iw}}{2} = \\ &= \frac{e^{iz}e^{iw} + e^{-iz}e^{-iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw}}{4} \\ \sin z \sin w &= \frac{e^{iz} - e^{-iz}}{2i}\frac{e^{iw} - e^{-iw}}{2i} = \\ &= \frac{-e^{iz}e^{iw} - e^{-iz}e^{-iw} + e^{-iz}e^{iw} + e^{iz}e^{-iw}}{4}\end{aligned}$$

so $\cos(z+w) = \cos z \cos w - \sin z \sin w$. In a similar way it is proved that $\sin(z+w) = \sin z \cos w + \cos z \sin w$.

3.2.8 Exercise

Of course

$$\tan z = \frac{\sin z}{\cos z} \tag{3.31}$$

is defined and analytic where $\cos z \neq 0$. Now $\cos z = 0$ means $e^{iz} + e^{-iz} = 0$, that is $e^{2iz} = -1$; by Exercise 6 point 2 that means $2iz = i(\pi + 2k\pi)$, and

$$z = \frac{\pi}{2} + k\pi. \tag{3.32}$$

3.2.9 Exercise

First, observe that if $z_n \to z$, then $|z_n| \to |z|$: it follows straight from the inequality $||z_n| - |z|| \leq |z_n - z|$. This yields that $r_n \to r$.

Now

$$z_n - z = re^{i\theta}\left(\frac{r_n}{r}e^{i(\theta_n - \theta)} - 1\right) \tag{3.33}$$

and since $z_n - z \to 0$, and $re^{i\theta} \neq 0$, then we must have $\frac{r_n}{r}e^{i(\theta_n-\theta)} - 1 \to 0$; but $\frac{r_n}{r} \to 1$, so $e^{i(\theta_n-\theta)} \to 1$, wich yields $(\theta_n - \theta) \to 2k\pi$ for some $k \in \mathbb{Z}$. Since by hypothesis $-\pi < \theta < \pi$ and $-\pi < \theta_n < \pi$, the only possibility is $(\theta_n - \theta) \to 0$, which means $\theta_n \to \theta$.

3.2.10 Exercise

As in Proof of Proposition 2.20, take $a \in G$, $s \in \mathbb{C}$ such that $s \neq 0$ and $a+s \in G$. Now $g(f(a)) = h(a)$, $g(f(a+s)) = h(a+s)$ and since h is injective and $s \neq 0$, we have $g(f(a+s)) \neq g(f(a))$ which yields $f(a+s) - f(a) \neq 0$. Now

$$\frac{g(f(a+s)) - g(f(a))}{s} = \frac{h(a+s) - h(a)}{s} \tag{3.34}$$

so

$$\frac{g(f(a+s)) - g(f(a))}{f(a+s) - f(a)} \cdot \frac{f(a+s) - f(a)}{s} = \frac{h(a+s) - h(a)}{s}; \tag{3.35}$$

since f is continuous, $f(a+s) \to f(a)$ as $s \to 0$, and this gives

$$\lim_{s\to 0} \frac{g(f(a+s)) - g(f(a))}{f(a+s) - f(a)} = g'(f(a)) \tag{3.36}$$

and since $g'(f(a)) \neq 0$ we finally get that

$$f'(a) = \lim_{s\to 0} \frac{f(a+s) - f(a)}{s} = \frac{h'(a)}{g'(f(a))}. \tag{3.37}$$

3.2.11 Exercise

Isn't it obvious? If f is a branch of the logarithm, then $e^{f(z)} = z$ for any $z \in G$. So $e^{nf(z)} = (e^{f(z)})^n = z^n$ for any $z \in G$.

3.2.12 Exercise

By definition

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} \tag{3.38}$$

where log is the principal branch of the logarithm, that is, if

$$G = \mathbb{C} - \{z \in \mathbb{C} \mid \mathfrak{Im}\, z \neq 0 \vee \mathfrak{Re}\, z > 0\} \tag{3.39}$$

then $\log : G \to \mathbb{C}$, and if $z = re^{i\theta}$ with $-\pi < \theta < \pi$ then $\log(z) = \log(r) + i\theta$. Then

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}(\log r + i\theta)} = \sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right) \tag{3.40}$$

and now

$$-\frac{\pi}{2} < \frac{\theta}{2} < \frac{\pi}{2} \tag{3.41}$$

so the real part of this complex number is positive.

3.2.13 Exercise

If log is the principal branch of the logarithm, and

$$f_k(z) = e^{\frac{1}{n}\log z} e^{\frac{2k\pi}{n}i} \quad k = 0, 1, \ldots, n-1 \quad z \in G \tag{3.42}$$

then

$$(f_k(z))^n = z \tag{3.43}$$

and the functions f_k are all distinct and analytic on G. On the other hand, if g and h are two any functios that satisfy the same conditions, then

$$F(z) = \frac{g(z)}{h(z)} \tag{3.44}$$

defines an analytic function on G, since $h(z) \neq 0$ on G. Then

$$F(z)^n = \left(\frac{g(z)}{h(z)}\right)^n = \frac{g(z)^n}{h(z)^n} = 1 \tag{3.45}$$

which yields that for any $z \in G$ there is an integer k such that $0 \leq k \leq n-1$ and

$$\frac{g(z)}{h(z)} = e^{\frac{2k\pi}{n}i}. \tag{3.46}$$

But F is continuous, so $\mathfrak{Im}\, F$ is connected, therefore there is one integer k that satisfies the last condition for all $z \in G$, otherwise $\mathfrak{Im}\, F$ would contain at least two distinct isolated points, and would not be connected. In particular, if f is any function satisfying the wanted conditions, then

$$\frac{f(z)}{e^{\frac{1}{n}\log z}} = e^{\frac{2k\pi}{n}i} \quad z \in G \tag{3.47}$$

for some $k = 0, 1, \ldots, n-1$, so $f = f_k$.

3.2.14 Exercise

Let

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) \tag{3.48}$$

then by hypothesis $v = 0$ in G. By the Cauchy-Riemann equations

$$u_x = v_y = 0$$
$$u_y = -v_x = 0$$

in G, and since G is connected, it follows that u, and then f, is constant.

3.2.15 Exercise

Of course, if $z \neq 0$

$$\exp\left(\frac{1}{z}\right) = \exp\left(\frac{1}{x+iy}\right) = \exp\left(\frac{x}{x^2+y^2}\right)\exp\left(-i\frac{y}{x^2+y^2}\right). \tag{3.49}$$

The function

$$f(x) = \frac{x}{x^2+y^2} \tag{3.50}$$

takes any real value k on the circle

$$\left(x - \frac{1}{2k}\right)^2 + y^2 - \frac{1}{4k^2} = 0 \tag{3.51}$$

and the function

$$f(x) = \frac{y}{x^2+y^2} \tag{3.52}$$

takes any real value h on the circle

$$\left(y - \frac{1}{2h}\right)^2 + x^2 - \frac{1}{4h^2} = 0. \tag{3.53}$$

All these circles go through $(0,0)$, which means that for any $r > 0$, in the set $\{0 < |z| < r\}$ the function $\exp(1/z)$ can take any possible value $re^{i\theta}$ with $r > 0$. That is, $A = \mathbb{C} - 0$.

3.2.16 Exercise

For instance

$$G = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z \neq 0 \vee \mathfrak{Re}\, z > 1 \vee \mathfrak{Re}\, z < -1\} \tag{3.54}$$

and

$$f(z) = \exp\left(\frac{\log(1-z^2)}{2}\right)$$
$$g(z) = \exp\left(\frac{\log(1-z^2)}{2} + \pi i\right).$$

To see that $f(z)^2 = g(z)^2 = 1 - z^2$ is trivial.

Now what can possibly mean that G is maximal? Maybe that if $H \supseteq G$, $h : H \to \mathbb{C}$ such that $h(z)^2 = 1 - z^2$ in H and for instance $h_{|G} = f$, then $H = G$. To prove this, suppose there is $z_0 \in H$ but $z_0 \notin G$. Then $z_0 = (x, 0)$ with either $x > 1$ or $x < -1$. Say $x > 1$. For $-\pi < \theta < \pi$ take

$$h(\theta) = \exp\left(\frac{\log\left((x^2-1)e^{i\theta} + 1\right)}{2}\right); \tag{3.55}$$

if $\theta \neq 0$ then $h(\theta) \in G$, $h(0) = x$ and h is obviously continuous. Now

$$f(h(\theta)) = \exp\left(\frac{\log(1 - h(\theta)^2)}{2}\right) = \exp\left(\frac{\log(-(x^2-1)e^{i\theta})}{2}\right) =$$
$$= \exp\left(\frac{\log((x^2-1)e^{i(\theta-\pi)})}{2}\right).$$

If $0 < \theta < \pi$ then $-\pi < \theta - \pi < 0$ so

$$f(h(\theta)) = \exp\left(\frac{1}{2}\left(\log(x^2-1) + i(\theta - \pi)\right)\right) \tag{3.56}$$

and

$$\lim_{\theta \to 0^+} h(\theta) = \exp\left(\frac{1}{2}\log(x^2-1) - \frac{\pi}{2}i\right) = \sqrt{x^2-1}\, e^{-\frac{\pi}{2}i} = -i\sqrt{x^2-1}. \tag{3.57}$$

If $-\pi < \theta < 0$ then $-2\pi < \theta - \pi < -\pi$ and $0 < \theta + \pi < \pi$ so

$$f(h(\theta)) = \exp\left(\frac{1}{2}\left(\log(x^2-1) + i(\theta+\pi)\right)\right) \tag{3.58}$$

and

$$\lim_{\theta\to 0^-} h(\theta) = \exp\left(\frac{1}{2}\log(x^2-1) + \frac{\pi}{2}i\right) = \sqrt{x^2-1}\,e^{\frac{\pi}{2}i} = i\sqrt{x^2-1}. \tag{3.59}$$

What we have shown is that h cannot be continuous in $(x, 0)$ if this point is in H but not in G, so if H is continuous G is maximal. As for analyticity, it follows from log's and exp's.

3.2.17 Exercise

Is it really an exercise? Well, $G = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z \neq 0 \vee \mathfrak{Re}\, z < 1\}$, $f : G \to \mathbb{C}$,

$$f(z) = \exp\left(\frac{1}{2}\log(1-z)\right). \tag{3.60}$$

3.2.18 Exercise

That doesn't seem much true. Say

$$f(z) = \exp(a\log(x))$$
$$g(z) = \exp(b(\log(x) + 2\pi i))$$

(by the way, here G needs to be connected) then

$$f(z)g(z) = \exp((a+b)\log(z) + b2\pi i) \tag{3.61}$$

and the latter deosn't look a bit like a branch of z^{a+b}.

For instance, take $a = b = \frac{1}{2}$:

$$f(z)g(z) = \exp\left(\left(\frac{1}{2}+\frac{1}{2}\right)\log(z) + \frac{1}{2}(2\pi i)\right) = ze^{\pi i} = -z \tag{3.62}$$

and there is no reasonable way to consider $-z$ a s a branch of $z^{\frac{1}{2}+\frac{1}{2}} = z^1 = z$.

Maybe we need to add the hypothesis that $f(z)$ and $g(z)$ are *the same* branch of z^a and z^b respectively, that is,

$$f(z) = \exp(a\,\mathrm{lg}(x))$$
$$g(z) = \exp(b\,\mathrm{lg}(x))$$

where lg is any branch of the logarithm on G. Now (even if G is not connected)

$$f(z)g(z) = \exp((a+b)\,\mathrm{lg}(z)) \tag{3.63}$$

and so fg is a branch of z^{a+b}. The same for f/g.

As well, let $\log : D \to \mathbb{C}$ be a branch of logarithm, $a \in \mathbb{C}$, $b \in \mathbb{C}$ and

$$\begin{aligned} f : D &\to \mathbb{C} \\ z &\mapsto \exp(a\log z) \end{aligned} \tag{3.64}$$

$$\begin{aligned} g : D &\to \mathbb{C} \\ z &\mapsto \exp(b\log z) \end{aligned} \tag{3.65}$$

so that f and g are branches of z^a and z^b respectively. To get a branch of z^{ab} it would seem fair enough to consider $g \circ f_{|f^{-1}(D)}$, but that is not the case. To see this, let $D = \{z \in \mathbb{C} \mid \mathfrak{Re}(z) > 0 \vee \mathfrak{Im}(z) \neq 0\}$, $\log : D \to \mathbb{C}$ be the principal branch of logrithm, $a = 2$, $b = \frac{1}{2}$ and $\bar{z} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \exp(i\frac{3}{4}\pi)$. Remark that if $z \in D$ then $\mathfrak{Im}(\log(z)) \in (-\pi, \pi)$. Clearly $\bar{z} \in D$, so

$$f(\bar{z}) = \exp(2\log(\bar{z})) = \exp\left(i\frac{3}{2}\pi\right) = -i. \tag{3.66}$$

Now $-i \in D$, so $\bar{z} \in f^{-1}(D)$, but $\log(-i) = -i\frac{\pi}{2}$ and

$$g(f(\bar{z})) = \exp\left(\frac{1}{2}\log(-i)\right) = \exp\left(-i\frac{\pi}{4}\right) = -\bar{z}. \tag{3.67}$$

But $ab = 1$, so $g \circ f_{|f^{-1}(D)}$ cannot be a branch of z^{ab}.

The proper restrition for f is to the set

$$E = \{z \in D \mid a\log(z) \in \text{Img}(\log)\} \tag{3.68}$$

that is, $z \in E \iff \mathfrak{Im}(a\log(z)) \in (-\pi, \pi)$. Let $z \in E$. By the definition of logarithm it follows that there exists a $k \in \mathbb{Z}$ such that

$$\log\left(\exp\left(a\log(z)\right)\right) = a\log(z) + i2k\pi; \tag{3.69}$$

but $\mathfrak{Im}\left(\log\left(\exp\left(a\log(z)\right)\right)\right) - \mathfrak{Im}\left(a\log(z)\right) < 2\pi$, so

$$\log\left(\exp\left(a\log(z)\right)\right) = a\log(z). \tag{3.70}$$

Then

$$g(f(z)) = \exp\left(b\log\left(\exp(a\log(z))\right)\right) = \exp(ab\log(z)) \tag{3.71}$$

which means that $g \circ f_{|E}$ is a branch of z^{ab}.

3.3 Analitic functions as mappings. Möbius transformations

3.3.1 Exercise

3.3.2 Exercise

3.3.3 Exercise

3.3.4 Exercise

3.3.5 Exercise

3.3.6 Exercise

(a) $7 + i$

(b) $\frac{-2i\cdot 2+2i}{-i\cdot 2+i\cdot(1+i)} = \frac{-2i}{-1-i} = \frac{2+2i}{2} = 1 + i$

(c) $\frac{-2i}{1-i} = \frac{2-2i}{2} = 1 - i$

(d) $\frac{1-i-1-i}{1-i} = \frac{-2i}{1-i} = 1 - i$

3.3.7 Exercise

Observe that T must be a Möbius transformation. It follows that $a = c$ implies $b \neq d$, $a = 0$ implies $b \neq 0$, and $c = 0$ implies $d \neq 0$. Now, allowing that $y/w = \infty$ if $w = 0$ and $z \neq 0$, we have

$$\begin{aligned} T(z) &= 1 \qquad \text{if} \quad z = \tfrac{b-d}{c-a} \\ T(z) &= 0 \qquad \text{if} \quad z = -\tfrac{b}{a} \\ T(z) &= \infty \qquad \text{if} \quad z = -\tfrac{d}{c} \end{aligned} \tag{3.72}$$

so

$$T(z) = \left(z, \frac{b-d}{c-a}, -\frac{b}{a}, -\frac{d}{c}\right). \tag{3.73}$$

3.3.8 Exercise

We know that

$$T(z) = \left(z, \frac{b-d}{c-a}, -\frac{b}{a}, -\frac{d}{c}\right). \tag{3.74}$$

Since $T(\mathbb{R}_\infty) = \mathbb{R}_\infty$ we have $\frac{b-d}{c-a} \in \mathbb{R}_\infty$, $-\frac{b}{a} \in \mathbb{R}_\infty$, $-\frac{d}{c} \in \mathbb{R}_\infty$. Then if $c - a \neq 0$, $a \neq 0$, $c \neq 0$

$$T(z) = \frac{\left(\frac{b-d}{c-a} + \frac{d}{c}\right) z - \left(\frac{d-b}{a-c} - \frac{d}{c}\right)\frac{b}{a}}{\left(\frac{b-d}{c-a} + \frac{b}{a}\right) z - \left(\frac{d-b}{a-c} - \frac{b}{a}\right)\frac{d}{c}} \tag{3.75}$$

if $c - a = 0$

$$T(z) = \frac{z + \frac{b}{a}}{z + \frac{d}{c}} \tag{3.76}$$

if $a = 0$

$$T(z) = \frac{\frac{b-d}{c-a} + \frac{d}{c}}{z + \frac{d}{c}} \tag{3.77}$$

if $c = 0$

$$T(z) = \frac{z + \frac{b}{a}}{\frac{b-d}{c-a} + \frac{b}{a}}. \tag{3.78}$$

3.3.9 Exercise

If T is a Möbius transformation, $|a|^2 + |b|^2 = |c|^2 + |d|^2$ and $a\bar{b} = c\bar{d}$ then $T(\Gamma) = \Gamma$. Indeed, if $|z| = 1$ then

$$|T(z)| = \left|\frac{az+b}{cz+d}\right| = \frac{|az+b|}{|cz+d|} = \frac{|az|^2 + |b|^2 + 2\,\mathfrak{Re}(az\bar{b})}{|cz|^2 + |d|^2 + 2\,\mathfrak{Re}(cz\bar{d})} =$$
$$= \frac{|a|^2\,|z|^2 + |b|^2 + 2\,\mathfrak{Re}(az\bar{b})}{|c|^2\,|z|^2 + |d|^2 + 2\,\mathfrak{Re}(cz\bar{d})} = \frac{|a|^2 + |b|^2 + 2\,\mathfrak{Re}(az\bar{b})}{|c|^2 + |d|^2 + 2\,\mathfrak{Re}(cz\bar{d})} = 1.$$

If $T(\Gamma) = \Gamma$ then T is a Möbius transformationand

$$|T(1)| = \left|\frac{a+b}{c+d}\right| = \frac{|a+b|}{|c+d|} = \frac{|a|^2 + |b|^2 + 2\,\mathfrak{Re}(a\bar{b})}{|c|^2 + |d|^2 + 2\,\mathfrak{Re}(c\bar{d})} = 1,$$
$$|T(-1)| = \left|\frac{-a+b}{-c+d}\right| = \frac{|-a+b|}{|-c+d|} = \frac{|a|^2 + |b|^2 - 2\,\mathfrak{Re}(a\bar{b})}{|c|^2 + |d|^2 - 2\,\mathfrak{Re}(c\bar{d})} = 1,$$
$$|T(i)| = \left|\frac{ia+b}{ic+d}\right| = \frac{|ia+b|}{|ic+d|} = \frac{|a|^2 + |b|^2 + 2\,\mathfrak{Re}(ia\bar{b})}{|c|^2 + |d|^2 + 2\,\mathfrak{Re}(ic\bar{d})} =$$
$$= \frac{|a|^2 + |b|^2 - 2\,\mathfrak{Im}(a\bar{b})}{|c|^2 + |d|^2 - 2\,\mathfrak{Im}(c\bar{d})} = 1$$
$$|T(-i)| = \left|\frac{-ia+b}{-ic+d}\right| = \frac{|-ia+b|}{|-ic+d|} = \frac{|a|^2 + |b|^2 + 2\,\mathfrak{Re}(-ia\bar{b})}{|c|^2 + |d|^2 + 2\,\mathfrak{Re}(-ic\bar{d})} =$$
$$= \frac{|a|^2 + |b|^2 + 2\,\mathfrak{Im}(a\bar{b})}{|c|^2 + |d|^2 + 2\,\mathfrak{Im}(c\bar{d})} = 1$$

whence

$$|a|^2 + |b|^2 + 2\,\mathfrak{Re}(a\bar{b}) = |c|^2 + |d|^2 + 2\,\mathfrak{Re}(c\bar{d}) \tag{3.79}$$
$$|a|^2 + |b|^2 - 2\,\mathfrak{Re}(a\bar{b}) = |c|^2 + |d|^2 - 2\,\mathfrak{Re}(c\bar{d}) \tag{3.80}$$
$$|a|^2 + |b|^2 - 2\,\mathfrak{Im}(a\bar{b}) = |c|^2 + |d|^2 - 2\,\mathfrak{Im}(c\bar{d}) \tag{3.81}$$
$$|a|^2 + |b|^2 + 2\,\mathfrak{Im}(a\bar{b}) = |c|^2 + |d|^2 + 2\,\mathfrak{Im}(c\bar{d}). \tag{3.82}$$

From 3.79 and 3.80 follows thet $|a|^2 + |b|^2 = |c|^2 + |d|^2$. Then from 3.79 and 3.80 follows that $\mathfrak{Re}(a\bar{b}) = \mathfrak{Re}(c\bar{d})$, from 3.81 and 3.82 that $\mathfrak{Im}(a\bar{b}) = \mathfrak{Im}(c\bar{d})$.

3.3.10 Exercise

Let $\gamma = \{z \in \mathbb{C} \mid |z| = 1\}$ and

$$T(z) = \frac{az+b}{cz+d}. \tag{3.83}$$

If T is a Möbius transformation and $T(D) = D$, then $T(\gamma) = \gamma$. Indeed, $T(\overline{D}) \subseteq \overline{T(D)} = \overline{D}$ since T is continuous; $T(\overline{D})$ is closed, since $\overline{D}$ is compact and therefore such is $T(\overline{D})$, so $T(\overline{D}) \supseteq \overline{T(D)} = \overline{D}$. Then $T(\overline{D}) = \overline{D}$. But $\overline{D} = \{z \in \mathbb{C} \mid |z| \leq 1\}$. The last two imply that $T(\gamma) = \gamma$. As seen in Exercise 3.3.9 this implies that $|a|^2 + |b|^2 = |c|^2 + |d|^2$ and $a\overline{b} = c\overline{d}$. Giving γ the orientation $C = (1, i, -1)$, 0 lies in the left of (γ, C):

$$(0, 1, i, -1) = \frac{\frac{0-i}{0+1}}{\frac{1-i}{1+i}} = 1 - i \tag{3.84}$$

so $T(0)$ must lie in the left of (γ, C) as well, that is $\mathfrak{Im}(T(0), 1, i, -1) < 0$. But

$$T(0) = \frac{b}{d} \tag{3.85}$$

and

$$\left(\frac{b}{d}, 1, i, -1\right) = \frac{\frac{\frac{b}{d}-i}{\frac{b}{d}+1}}{\frac{1-i}{1+1}} = 1 + i\frac{b-d}{b+d} \tag{3.86}$$

so

$$\mathfrak{Re}\left(\frac{b-d}{b+d}\right) < 0. \tag{3.87}$$

But

$$\frac{b-d}{b+d} = \frac{|b|^2 - d\overline{b} + b\overline{d} - |d|^2}{|b+d|^2} = \frac{|b|^2 - |d|^2 + 2\,\mathfrak{Im}(b\overline{d})i}{|b+d|^2} \tag{3.88}$$

so 3.87 yields $|d| > |b|$.

If T is a Möbius transformation and $|a|^2 + |b|^2 = |c|^2 + |d|^2$, $a\overline{b} = c\overline{d}$ and $|d| > |b|$, then $T(\gamma) = \gamma$ and $\mathfrak{Im}(T(0), 1, i, -1) < 0$, so $T(D) = D$.

3.3.11 Exercise

3.3.12 Exercise

3.3.13 Exercise

3.3.14 Exercise

Let c_1 and c_2 be the centres of the two circles γ_1 and γ_2. Then a, c_1, c_2 are aligned and distinct. That is, $c_1 - a = \alpha(c_2 - a)$ where $\alpha \in \mathbb{R}$. Moreover, $\alpha > 0$. We can suppose also $\alpha > 1$, by swapping names between c_1 and c_2 if needed.

If

$$\begin{aligned} c_1 - a &= |c_1 - a|\, e^{i\theta} \\ c_2 - a &= |c_2 - a|\, e^{i\tau} \\ c_1 - c_2 &= |c_1 - c_2|\, e^{i\sigma} \end{aligned}$$

then

$$|c_1 - a|\, e^{i\theta} = \alpha\, |c_2 - a|\, e^{i\tau} \tag{3.89}$$

and

$$\frac{|c_1 - a|}{\alpha\, |c_1 - a|} = e^{i(\tau - \theta)}. \tag{3.90}$$

Since the left side is real and positive, so must the right side be too, which implies $\tau = \theta$. Also

$$c_1 - c_2 = c_1 - a - (c_2 - a) = (\alpha - 1)(c_2 - a) \tag{3.91}$$

where $\alpha - 1 > 0$, so

$$\frac{|c_1 - c_2|}{(\alpha - 1)\,|c_2 - a|} = e^{i(\sigma - \tau)} \tag{3.92}$$

and $\sigma = \tau$.

The translation

$$T_1(z) = z - a \tag{3.93}$$

takes a in 0. The rotation

$$T_2(z) = e^{-i\sigma} z \tag{3.94}$$

takes $T_1(c_1)$ and $T_1(c_2)$ on the imaginary axis. In fact

$$\begin{aligned} T_2(T_1(c_1)) &= e^{-i\sigma}(c_1 - a) = e^{-i\sigma}\,|c_1 - a|\,e^{i\theta} = e^{-i\sigma}\,|c_1 - a|\,e^{i\sigma} = |c_1 - a|\,i \\ T_2(T_1(c_2)) &= e^{-i\sigma}(c_2 - a) = e^{-i\sigma}\,|c_2 - a|\,e^{i\tau} = e^{-i\sigma}\,|c_2 - a|\,e^{i\sigma} = |c_1 - a|\,i. \end{aligned}$$

Let

$$\begin{aligned} d_1 &= |c_1 - a|\,i \\ d_2 &= |c_1 - a|\,i. \end{aligned}$$

Since for any z, w in $\mathbb{C}$ $|T_2(T_1(z)) - T_2(T_1(w))| = |z - w|$, $T_2 \circ T_1$ takes γ_1 onto

$$\Gamma_1 = \{z \in \mathbb{C} \mid \ |z - d_1| = |c_1 - a|\}$$

and γ_2 onto

$$\Gamma_2 = \{z \in \mathbb{C} \mid \ |z - d_2| = |c_2 - a|\}.$$

The inversion

$$T_3(z) = \frac{1}{z} \tag{3.95}$$

takes the circles Γ_1 and Γ_2 onto the straight lines $r_1 : \mathfrak{Im}\, z = -\frac{1}{2|c_1 - a|}$ and $r_2 : \mathfrak{Im}\, z = -\frac{1}{2|c_2 - a|}$ respectively. So $T_3 \circ T_2 \circ T_1$ sends γ_1 to r_1 and γ_2 to r_2. Moreover, D is mapped onto the stripe between r_1 and r_2.

Let

$$\begin{aligned} K &= \frac{1}{4}\left(\frac{1}{|c_1 - a|} + \frac{1}{|c_2 - a|}\right) \\ L &= \frac{1}{2}\left(\frac{1}{|c_2 - a|} - \frac{1}{|c_1 - a|}\right). \end{aligned}$$

The translation

$$T_4(z) = z + K \tag{3.96}$$

takes the stripe

$$\left\{z \in \mathbb{C} \mid -\frac{1}{2\,|c_2 - a|} \leq \mathfrak{Im}\, z \leq -\frac{1}{2\,|c_1 - a|}\right\} \tag{3.97}$$

onto the stripe

$$\left\{z \in \mathbb{C} \mid -\frac{L}{2} \leq \mathfrak{Im}\, z \leq \frac{L}{2}\right\} \tag{3.98}$$

and the dilation

$$T_5(z) = \frac{\pi}{L} \tag{3.99}$$

takes this last to the stripe

$$\left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} \le \mathfrak{Im}\, z \le \frac{\pi}{2} \right\}. \tag{3.100}$$

Eventually the funcion

$$T_6(z) = \frac{e^z - 1}{e^z + 1} \tag{3.101}$$

takes tha last stripe onto the unit disk. Let $T = T_6 \circ T_5 \circ T_4 \circ T_3 \circ T_2 \circ T_1$. The map T takes D onto the unit circle, and

$$T(z) = \frac{e^{\left[\frac{\pi}{L}\left(\frac{1}{e^{-i\sigma}(z-a)} + K\right)\right] - 1}}{e^{\left[\frac{\pi}{L}\left(\frac{1}{e^{-i\sigma}(z-a)} + K\right)\right] + 1}} \tag{3.102}$$

or, in the expanded form

$$T(z) = \frac{e^{\left[\frac{\pi}{\frac{1}{2}\left(\frac{1}{|c_2-a|} - \frac{1}{|c_1-a|}\right)}\left(\frac{1}{e^{-i\arg(c_1-c_2)}(z-a)} + \frac{1}{4}\left(\frac{1}{|c_1-a|} + \frac{1}{|c_2-a|}\right)\right)\right]} - 1}{e^{\left[\frac{\pi}{\frac{1}{2}\left(\frac{1}{|c_2-a|} - \frac{1}{|c_1-a|}\right)}\left(\frac{1}{e^{-i\arg(c_1-c_2)}(z-a)} + \frac{1}{4}\left(\frac{1}{|c_1-a|} + \frac{1}{|c_2-a|}\right)\right)\right]} + 1}. \tag{3.103}$$

3.3.15 Exercise

Let

$$D = \{ z \in \mathbb{C} \mid |z| < 1 \}, \tag{3.104}$$

$$H = \{ z \in \mathbb{C} \mid \mathfrak{Re}\, z < 0 \} \tag{3.105}$$

and

$$E = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}. \tag{3.106}$$

The map

$$T(z) = \frac{z-1}{z+1} \tag{3.107}$$

maps D onto H. Then

$$S(z) = e^{\frac{z-1}{z+1}} \tag{3.108}$$

maps D onto E.

Let's see some properties of S. Let $h < 0$. We have

$$T\left(\frac{h}{1-h} + \frac{1}{1-h} e^{i\theta}\right) = h + i\frac{(1-h)\sin\theta}{1+\cos\theta} \tag{3.109}$$

which means that S maps the circle C_h^1 centered in

$$a = \frac{h}{1-h} \tag{3.110}$$

with radius

$$r = \frac{1}{1-h} \tag{3.111}$$

minus the point -1 into the circle C_h^2 centered in 0 with radius e^h. Moreover

$$\lim_{\theta \to \pi^-} \frac{\sin\theta}{1+\cos\theta} = +\infty \tag{3.112}$$

so when the point z approaches -1 on the upper half of the circle C_h^1 its image $S(z)$ spans infinitely many times the circle C_h^2 turning counterclockwise around 0. Also if the point z approaches -1 on the lower half of the circle C_h^1 its image $S(z)$ spans infinitely many times the circle C_h^2 turning clockwise around 0.

3.3.16 Exercise

The Möbius transformation

$$T(z) = \frac{z-1}{z+1}$$

tekes G onto $\mathbb{C} - (\{z \in \mathbb{C} \mid \mathfrak{Im}(z) = 0, \mathfrak{Re}(z) \leq 0\} \cup \{1\})$. Therefore, if log is the principal branch of the logarithm, the function

$$g(z) = \log(T(z))$$

takes G onto $\{z \in \mathbb{C} \mid |\mathfrak{Im}(z)| < \pi\} - \{0\}$. Then

$$h(z) = \exp\left(\frac{1}{2}\log(T(z))\right)$$

takes G onto $H = \{z \in \mathbb{C} \mid \mathfrak{Re}(z) > 0\} - \{1\}$. If $\sqrt{\ }$ means the principal branch of the square root, then h can be written as

$$h(z) = \sqrt{\frac{z-1}{z+1}}.$$

The Möbius transformation

$$S(w) = \frac{2w-1}{2w+1}$$

takes H onto $\{z \in \mathbb{C} \mid |z| < 1\} - \left\{\frac{1}{3}\right\}$, so S^2 takes H onto the unit circle. Summing up, the analytic function

$$f(z) = \left(\frac{2\sqrt{\frac{z-1}{z+1}} - 1}{2\sqrt{\frac{z-1}{z+1}} + 1}\right)^2$$

takes G onto the unit circle.

3.3.17 Exercise

Let $f(G) \subseteq \Gamma$ where Γ is a circle. There is a Möbius transformation T such that $T(\Gamma) = \mathbb{R}_\infty$, then $T \circ f(G) \subseteq \mathbb{R}_\infty$, which implies that $T \circ g$ is constant, and so is f.

3.3.18 Exercise

3.3.19 Exercise

3.3.20 Exercise

Let

$$\alpha = \lambda a$$
$$\beta = \lambda b$$
$$\gamma = \lambda c$$
$$\delta = \lambda d.$$

Then for $z \in \mathbb{C}$

$$T(z) = \frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az+b}{cz+d} = S(z). \tag{3.113}$$

Let $S = T$. From

$$T^{-1}(w) = \frac{\delta w - \beta}{-\gamma w + \alpha} \tag{3.114}$$

follows from all $w \in \mathbb{C}$

$$ST^{-1}(w) = \frac{a\frac{\delta w-\beta}{-\gamma w+\alpha} + b}{c\frac{\delta w-\beta}{-\gamma w+\alpha} + d} = \frac{(a\delta - b\gamma)w + b\alpha - a\beta}{(c\delta - d\gamma)w + d\alpha - c\beta} = w \tag{3.115}$$

whence

$$(c\delta - d\gamma)w^2 + (d\alpha - c\beta - a\delta + b\gamma)w - b\alpha + a\beta = 0. \tag{3.116}$$

This implies

$$\begin{aligned} c\delta - d\gamma &= 0 \\ d\alpha - c\beta - a\delta + b\gamma &= 0 \\ -b\alpha + a\beta &= 0. \end{aligned}$$

whence

$$\begin{aligned} \frac{c}{\gamma} &= \frac{d}{\delta} \\ \frac{a}{\alpha} &= \frac{b}{\beta}. \end{aligned}$$

Now let

$$\begin{aligned} t_1 &= \frac{c}{\gamma} = \frac{d}{\delta} \\ t_2 &= \frac{a}{\alpha} = \frac{b}{\beta}. \end{aligned}$$

Then

$$t_1\delta\alpha - t_1\gamma\beta - t_2\alpha\delta + t_2\beta\gamma = 0 \tag{3.117}$$

and

$$(t_1 - t_2)(\alpha\delta - \beta\gamma) = 0 \tag{3.118}$$

which implies, if T is a Möbius transformation, that $t_1 = t_2$ and so

$$\begin{aligned} \alpha &= t_1^{-1}a \\ \beta &= t_1^{-1}b \\ \gamma &= t_1^{-1}c \\ \delta &= t_1^{-1}d. \end{aligned}$$

If T is not a Möbius transformation, then it has constant value $\frac{\beta}{\delta}$, so it must be $\alpha = \gamma = 0$, otherwise T wouldn't be defined for $z = -\frac{\delta}{\gamma}$. From $T = S$ follows that S also must have constant value $\frac{\beta}{\delta}$, but if S is constant its value is $\frac{b}{d}$ and $a = c = 0$, so

$$\frac{b}{d} = \frac{\beta}{\delta} \tag{3.119}$$

and

$$\frac{b}{\beta} = \frac{d}{\delta}. \tag{3.120}$$

3.3.21 Exercise

Well:

$$\begin{aligned} S^{-1}TS(S^{-1}(z_1)) &= S^{-1}(T(z_1)) = S^{-1}(z_1) \\ S^{-1}TS(S^{-1}(z_2)) &= S^{-1}(T(z_2)) = S^{-1}(z_2). \end{aligned}$$

3.3.22 Exercise

Let T be defined by

$$T(z) = \frac{az+b}{cz+d}. \tag{3.121}$$

(a) If T has 0 and ∞ as its only fixed points, then

$$\frac{b}{d} = 0$$
$$\frac{a}{c} = \infty$$

whence

$$b = 0$$
$$c = 0$$

and

$$T(z) = \frac{a}{d}z. \tag{3.122}$$

with $a \neq d$, and it is clear that such a transformation has 0 and ∞ as its only fixed points.

(b) If T has ∞ as its only fixed point, then $\frac{a}{c} = \infty$ and $c = 0$. Then

$$T(z) = \frac{az + b}{d}. \tag{3.123}$$

Now z is a fixed point of T if and only if

$$\left(\frac{a}{d} - 1\right) z + \frac{b}{d} = 0 \tag{3.124}$$

so T has no other fixed point if and only if $a = d$ and $b \neq 0$, that is:

$$T(z) = z + \frac{b}{d}. \tag{3.125}$$

3.3.23 Exercise

Let T be defined by

$$T(z) = \frac{az + b}{cz + d}. \tag{3.126}$$

If $T(0) = \infty$ and $T(\infty) = 0$ then

$$\frac{b}{d} = \infty$$
$$\frac{a}{c} = 0$$

so $d = 0$ and $a = 0$, and

$$T(z) = \frac{b}{cz}. \tag{3.127}$$

If $T(z) = az^{-1}$ it is obvious that $T(0) = \infty$ and $T(\infty) = 0$.

3.3.24 Exercise

If T has one fixed point z_1, let R be a Möbius transformation such that $R(\infty) = z_1$. Then $R^{-1}TR$ has ∞ as its only fixed point. Indeed, $R^{-1}TR(\infty) = R^{-1}T(z_1) = R^{-1}(z_1) = \infty$, and from $R^{-1}TR(z) = z$ follows $TR(z) = R(z)$, so $R(z)$ is a fixed point of T, that is $R(z) = z_1$, and $z = \infty$. Then $R^{-1}TR$ is a translation, and so is $R^{-1}SR$. Since two translations commute

$$R^{-1}TRR^{-1}SR = R^{-1}SRR^{-1}TR \tag{3.128}$$

that is

$$R^{-1}TSR = R^{-1}STR \tag{3.129}$$

whence $TS = ST$.
If T has two fixed points z_1, z_2, let R be a Möbius transformationsuch that $R(\infty) = z_1$ and $R(0) = z_2$. Then $R^{-1}TR$ and $R^{-1}SR$ have fixed points ∞ and 0, so they are dilations, and since dilations commute, it follows as in the former case that S and T commute too.

3.3.25 Exercise

Let $\mathcal{M}$ be the group of Möbius transformations. For $u, v \in \mathbb{C}$, let

$$\mathcal{H}_{u,v} = \{T \in \mathcal{M} \mid T(u) = u, T(v) = v\}\,. \tag{3.130}$$

Then

- $T, S \in \mathcal{H}_{u,v} \Rightarrow TS^{-1} \in \mathcal{H}_{u,v}$
- $T, S \in \mathcal{H}_{u,v} \Rightarrow TS = ST$

that is, $\mathcal{H}_{u,v}$ is an abelian subgroup of $\mathcal{M}$.

Every $\mathcal{H}_{u,v}$ is maximal, that is, if $\mathcal{G}$ is a subgroup of $\mathcal{M}$ and $\mathcal{H}_{u,v} \subset \mathcal{G}$ then $\mathcal{G}$ is not abelian.

Indeed, suppose $u \neq v$ and let $R \in \mathcal{G}$ and $R \notin \mathcal{H}_{u,v}$. If $R(u) = u$ but $R(v) \neq v$ then R does not commute with any $T \in \mathcal{H}_{u,v}$: $TR(v) = RT(v) = R(v)$ implies that $R(v)$ is a fixed point of T, that is either $R(v) = v$ or $R(v) = u$, both cases impossible. The same if $R(v) = v$ but $R(u) \neq u$. If $R(u) \neq u$ and $R(v) \neq v$ then $RT = TR$ implies $RT(u) = TR(u) = R(u)$, that is $R(u)$ is a fixed point of T, so $R(u) = v$, and for the same reason $R(v) = u$. But R must have two fixed points, since if z is a fixed point of R then $z \neq u$ and $z \neq v$ but $RT(z) = TR(z) = T(z)$, so $T(z)$ is a fixed point of R and $T(z) \neq z$ because z is not a fixed point of T. Let w be the second fixed point of R. For the same reason as before, $T(z) = w$ and $T(w) = z$. But in $\mathcal{H}_{u,v}$ there is only one element which swaps points in $\mathbb{C}$ (see Proposition 8.3.4), so R cannot commute with all the elements of $\mathcal{H}_{u,v}$.

Now suppose $u = v$ and let $R \in \mathcal{G}$ and $R \notin \mathcal{H}_{u,v} = \mathcal{H}_{u,u}$, $T \in \mathcal{H}_{u,u}$. If R has one fixed point z then $z \neq u$ and $RT = TR$ implies $RT(u) = TR(u) = R(u)$, so $R(u) = u$ against the hypothesis. If R has two fixed points, z and w, suppose $z = u$ and $w \neq u$. Then $TR = RT$ implies $RT(w) = TR(w) = T(w)$ and $T(w)$ is a fixed point of R. But $T(w) = w$ implies $w = u$, $T(w) = z$ implies $T(w) = u$ and again $w = u$, against the hypothesis. The same if $z \neq u$ and $w = u$. If $z \neq u$ and $w \neq u$, then $TR = RT$ implies $RT(u) = TR(u) = R(u)$, that is $R(u) = u$, and $u = z$ or $u = w$, against the hypothesis.

Now, let $\mathcal{A}$ be an abelian subgroup of $\mathcal{M}$.

If $\mathcal{A}$ contains only elements with the same fixed points or point, then $\mathcal{A} \subseteq \mathcal{H}_{u,v}$ for some $u, v \in \mathbb{C}$, either $v \neq u$ or $u = v$.

If $\mathcal{A}$ contains at least two elements T and S with different fixed points z_1, z_2, and w_1, w_2, we have seen that $z_1 \neq z_2$, $w_1 \neq w_2$ and $\{z_1, z_2\} \cap \{w_1, w_2\} = \emptyset$. Furthermore, $T(w_1) = w_2$ and $S(z_1) = z_2$, that is, $T^2 = S^2 = I$. Now in $\mathcal{A}$ there must be $U = ST$ too, and $U \neq S$, $U \neq T$, since, for instance, $ST = S$ implies $T = I$, but in $\mathcal{A}$ there cannot be any other element of $\mathcal{M}$. Indeed, if R is any Möbius transformation that commutes with S and T, then $R^2 = I$ and each one of S, T, R must swap the others' fixed points, so STR has 6 fixed points, that is $STR = I$ and $R = ST = U$.

Summing up: if $\mathcal{A}$ is an abelian subgroup of $\mathcal{M}$ then either $\mathcal{A} \subseteq \mathcal{H}_{u,v}$ or for some elements $u, v \in \mathbb{C}$ with $u = v$ or $u \neq v$, or $\mathcal{A} = \{I, S, T, U\}$ where S, T, U are elements of $\mathcal{M}$ such that $S^2 = T^2 = U^2 = I$.

3.3.26 Exercise

(a) We have, for $z \in \mathbb{C}$

$$\phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}\right)(z) = \phi\left(\begin{pmatrix} a'a'' + b'c'' & a'b'' + b'd'' \\ c'a'' + d'c'' & c'b'' + d'd'' \end{pmatrix}\right)(z) = \tag{3.131}$$

$$= \frac{(a'a'' + b'c'')z + a'b'' + b'd''}{(c'a'' + d'c'')z + c'b'' + d'd''} \tag{3.132}$$

and

$$\phi\left(\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}\right) \circ \phi\left(\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}\right)(z) = \frac{a'\frac{a''z+b''}{c''z+d''} + b'}{c'\frac{a''z+b''}{c''z+d''} + d'} \tag{3.133}$$

$$= \frac{(a'a'' + b'c'')z + a'b'' + b'd''}{(c'a'' + d'c'')z + c'b'' + d'd''}. \tag{3.134}$$

If for all $z \in \mathbb{C}$

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = z \tag{3.135}$$

then for all $z \in \mathbb{C}$

$$z^2 + (d - a)z - b = 0 \tag{3.136}$$

whence

$$\begin{cases} c = 0 \\ b = 0 \\ a = d \end{cases} \tag{3.137}$$

and

$$\operatorname{Ker}\phi = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbb{C} \right\}. \tag{3.138}$$

It is obvious that $\operatorname{Img}\phi = \mathcal{M}$.

(b) We only need show that for any $A \in GL_2(\mathbb{C})$ there is $A' \in SL_2(\mathbb{C})$ such that $\phi(A') = \phi(A)$. Let $\Delta \in \mathbb{C}$ such that $\Delta^2 = \det A$, and

$$A' = \frac{1}{\Delta} A. \tag{3.139}$$

Clearly $\phi(A') = \phi(A)$ and $\det A' = \frac{1}{\Delta^2} \det A = 1$. Finally

$$\operatorname{Ker}\phi \cap SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{3.140}$$

3.3.27 Exercise

The group $\mathcal{M}$ of all Möbius transformation is simple because $SL_2(\mathbb{C})$ has no normal subgroups other than

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \tag{3.141}$$

which is the kernel of a surjective omomorphism from $SL_2(\mathbb{C})$ to $\mathcal{M}$.

3.3.28 Exercise

3.3.29 Exercise

(a) Since

$$u^{-1}_{\gamma,\delta}(z) = \frac{\overline{\gamma} z + \overline{\delta}}{-\delta z + \gamma}$$

we have

$$u_{\alpha,\beta} \circ u^{-1}_{\gamma,\delta}(z) = \frac{(\alpha\overline{\gamma} - \delta\beta)z + \alpha\overline{\delta} + \beta\gamma}{-(\overline{\beta}\overline{\gamma} + \delta\overline{\alpha})z - \overline{\beta}\,\overline{\delta} + \gamma\overline{\alpha}}$$

so $u_{\alpha,\beta} \circ u^{-1}_{\gamma,\delta} \in U$.

(b) Since $A \in SU_2 \iff \overline{A}^t = A^{-1} \wedge \det A = 1$, if $A, B \in SU_2$ we have

$$\overline{(AB^{-1})}^t = \overline{B^{-1}}^t \overline{A}^t = \overline{B}^{t^{-1}} \overline{A}^t = BA^{-1} = (AB^{-1})^{-1}$$

and

$$\det(AB) = \det A \det B = 1$$

so $AB \in SU_2$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then

$$\overline{A}^t = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$$

so $A \in SU_2$ implies

$$\begin{aligned} &|a|^2 + |b|^2 = 1 \\ &|c|^2 + |d|^2 = 1 \\ &a\overline{c} + b\overline{d} = 0 \\ &ad - bc = 1 \end{aligned}$$

From the last two

$$a = -\frac{\overline{d}}{\overline{c}}b, \quad b(|d|^2 + |c|^2) = -\overline{c}$$

and $b = -\overline{c}$. Again from the last but one

$$b(\overline{d} - a) = 0.$$

If $b \neq 0$ then $a = \overline{d}$. If $b = 0$ then also $c = 0$, $|a|^2 = |d|^2 = 1$ and $ad = 1$, so $a = d = 1$ or $a = d = -1$.

(c) This is false, since

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & \delta \\ -\overline{\delta} & \overline{\gamma} \end{pmatrix} = \begin{pmatrix} \alpha\gamma - \beta\overline{\delta} & \alpha\delta + \beta\overline{\gamma} \\ -\gamma\overline{\beta} - \overline{\alpha}\overline{\delta} & \delta\overline{\beta} + \overline{\alpha\gamma} \end{pmatrix}$$

while

$$u_{\alpha,\beta} \circ u_{\gamma,\delta}(z) = \frac{(\alpha\gamma - \delta\overline{\beta})z - \alpha\overline{\delta} - \overline{\beta}\overline{\gamma}}{(\beta\gamma + \delta\overline{\alpha})z - \beta\overline{\delta} + \overline{\alpha\gamma}}.$$

On the other hand the homomorphism $\phi : GL_2(\mathbb{C}) \to \mathcal{M}$ obviously takes $SU_2(\mathbb{C})$ onto U. From $I \in SU_2(\mathbb{C})$ and $-I \in SU_2(\mathbb{C})$ follows that $\ker \phi_{|SU_2(\mathbb{C})} = \{I, -I\}$.

(c) This way $T_v^{(l)}$ is *not* well defined. Take $v(z) = z$ and $f(z) = z$. Then $v = u_{1,1}$ and $v = u_{-1,-1}$. In the first case

$$T_v^{(l)}(f)(z) = (1)^1 f(v(z)) = z$$

in the second one

$$T_v^{(l)}(f)(z) = (-1)^1 f(v(z)) = -z.$$

The proper way is to take $l \in \mathbb{N}$. Since $\ker \phi_{|SU_2(\mathbb{C})} = \{I, -I\}$, if $V = u_{\alpha,\beta} = u_{\gamma,\delta}$ then

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & -\overline{\delta} \\ \delta & \overline{\gamma} \end{pmatrix}^{-1} = I$$

and $\alpha = \gamma$, $\beta = \delta$, or

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \gamma & -\overline{\delta} \\ \delta & \overline{\gamma} \end{pmatrix}^{-1} = -I$$

and $\alpha = -\gamma$, $\beta = -\delta$. Since

$$(\beta z + \overline{\alpha})^{2l} f(v(z)) = (-(\beta z + \overline{\alpha}))^{2l} f(v(z))$$

$T_u^{(l)}$ is well defined.

Then, if $v \in U$, say $v = u_{\alpha,\beta}$, so, for $f, g \in H_l$ and $a, b \in \mathbb{C}$:

$$\begin{aligned} T_v^{(l)}(af + bg)(z) &= (\beta z + \overline{\alpha})^{2l}(af(v(z)) + bg(v(z))) = \\ &= a(\beta z + \overline{\alpha})^{2l} f(v(z)) + b(\beta z + \overline{\alpha})^{2l} g(v(z)) = \\ &= aT_v^{(l)}(f)(z) + bT_v^{(l)}(g)(z) = \\ &= (aT_v^{(l)}(f) + bT_v^{(l)}(g))(z) \end{aligned}$$

so $T_v^{(l)}$ is a linear transformation. It is easy to check that for any $v \in U$ $T_v^{(l)} \circ T_{v^{-1}}^{(l)} = \mathrm{id}_{H_l}$. Let $\mathcal{A}(H_l)$ the group of automorphisms of H_l. The map

$$\begin{aligned} \psi : U &\to \mathcal{A}(H_l) \\ u &\mapsto T_u^{(l)} \end{aligned}$$

is *not* a homomorphisms, at least if the composition in $\mathcal{A}(H_l)$ is defined in the usual way. In fact, if $v, w \in U$ and $v = u_{\alpha,\beta}$, $w = u_{\gamma,\delta}$

$$T_{uv}^{(l)}(f)(z) = ((\beta\gamma + \overline{\alpha}\delta)z - \beta\overline{\delta} + \overline{\alpha\gamma})^{2l} f(uv(z))$$

and

$$\begin{aligned} (T_u^{(l)} \circ T_v^{(l)})(f)(z) &= T_u^{(l)}(T_v^{(l)}(f))(z) = (\beta z + \overline{\alpha})^{2l}(T_v^{(l)}(f))(u(z)) = \\ &= (\beta z + \overline{\alpha})^{2l}(\delta u(z) + \overline{\gamma})^{2l} f(v(u(z))) = \\ &= ((\alpha\delta + \beta\overline{\gamma})z - \delta\overline{\beta} + \overline{\alpha\gamma})^{2l} f(v(u(z))). \end{aligned}$$

But if we define the composition in $\mathcal{A}(H_l)$ as $T \circ S(f) = S(T(f))$, then

$$\begin{aligned} (T_u^{(l)} \circ T_v^{(l)})(f)(z) &= T_v^{(l)}(T_u^{(l)}(f))(z) = (\delta z + \overline{\gamma})^{2l}(T_u^{(l)}(f))(v(z)) = \\ &= (\delta z + \overline{\gamma})^{2l}(\beta v(z) + \overline{\alpha})^{2l} f(u(v(z))) = \\ &= ((\beta\gamma + \overline{\alpha}\delta)z - \beta\overline{\delta} + \overline{\alpha\gamma})^{2l} f(uv(z)). \end{aligned}$$

To show that ψ is injective, suppose $\psi(v)$ is the identity of $\mathcal{A}(H_l)$,that is, for any $f \in H_l$:

$$T_v^l(f) = f.$$

If $f(z) = 1$, and $v = u_{\alpha,\beta}$ then

$$T_v^l(f)(z) = (\beta z + \overline{\alpha})^{2l} f(v(z)) = (\beta z + \overline{\alpha})^{2l} = 1.$$

Since $2l$ is an even non-negative integer

$$\beta z + \overline{\alpha} = \pm 1$$

whence $\beta = 0$ and $\alpha = \pm 1$. That is, v is the identity of U.

3.3.30 Exercise

We have

$$|f(z)| = e^{\mathfrak{Re}\left(-\frac{i}{2}\log\left(\frac{iz+i}{-z+1}\right)\right)} = e^{\frac{1}{2}\mathfrak{Im}\left(\log\left(\frac{iz+i}{-z+1}\right)\right)} = e^{\frac{1}{2}\arg\left(\frac{iz+i}{-z+1}\right)}.$$

The Möbius transformation

$$T(z) = \frac{iz+i}{-z+1}$$

takes D onto the upper half plane, as

$$\begin{cases} T(1) = \infty \\ T(i) = -1 \\ T(-1) = 0. \end{cases}$$

This yields

$$0 < \arg\left(\frac{iz+i}{-z+1}\right) < \pi$$

whence

$$1 < |f(z)| < e^{\frac{\pi}{2}}.$$

If S is a Möbius transformation that maps D onto D and such that $f(S(z)) = f(z)$, then there is an integer k such that for every $z \in D$

$$-\frac{i}{2}\log\left(\frac{iS(z)+i}{-S(z)+1}\right) = -\frac{i}{2}\log\left(\frac{iz+i}{-z+1}\right) + 2k\pi i.$$

Since both the arguments of the logarithms belong to the upper half plane

$$\frac{i}{2}\log\left(\frac{S(z)+1}{-S(z)+1}\frac{-z+1}{z+1}\right) = 2h\pi$$

for some integer h, or

$$\frac{S(z)+1}{-S(z)+1}\frac{-z+1}{z+1} = e^{4h\pi}$$

which yields

$$S(z) = \frac{(e^{4k\pi}+1)z + e^{4k\pi} - 1}{(e^{4k\pi}-1)z + e^{4k\pi} + 1}.$$

All these Möbius transformations take D onto D.

Chapter 4

Complex Integration

4.1 Riemann-Stieltjes integrals

4.1.1 Exercise

If $P = \{t_0, \cdots, t_n\}$ is a partition of $[a, b]$, we have, since γ is non decreasing

$$\begin{aligned} V(\gamma; P) &= \sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| = \sum_{k=1}^{n} \gamma(t_k) - \gamma(t_{k-1}) = \\ &= \gamma(t_n) - \gamma(t_0) = \gamma(b) - \gamma(a). \end{aligned}$$

4.1.2 Exercise

(a) Say $\#\mathcal{Q} = \#\mathcal{P} + 1$. Then if

$$\mathcal{P} = \{t_0, \ldots, t_n\} \tag{4.1}$$

we will have, for some j such that $0 \le j \le n - 1$

$$\mathcal{Q} = \{t_0, \ldots, t_j, s, t_{j+1}, \ldots t_n\}; \tag{4.2}$$

then

$$\begin{aligned} v(\gamma; \mathcal{Q}) = &\sum_{k=1}^{j} |\gamma(t_k) - \gamma(t_{k-1})| + |\gamma(s) - \gamma(t_j)| + \\ &+ |\gamma(t_{j+1}) - \gamma(s)| + \sum_{k=j+2}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \end{aligned}$$

and

$$\begin{aligned} v(\gamma; \mathcal{P}) = &\sum_{k=1}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \le \\ &\le \sum_{k=1}^{j} |\gamma(t_k) - \gamma(t_{k-1})| + |\gamma(t_{j+1}) - \gamma(t_j)| + \\ &\quad + \sum_{k=j+2}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \le \\ &\le \sum_{k=1}^{j} |\gamma(t_k) - \gamma(t_{k-1})| + |\gamma(s) - \gamma(t_j)| + \\ &+ |\gamma(t_{j+1}) - \gamma(s)| + \sum_{k=j+2}^{n} |\gamma(t_k) - \gamma(t_{k-1})| = v(\gamma; \mathcal{Q}). \end{aligned}$$

If $\#\mathcal{Q} = \#\mathcal{P} + h$, then $\mathcal{Q} = \mathcal{P} \cup \{s_1, \ldots, s_h\}$.
Let

- $\mathcal{Q}_1 = \mathcal{P} \cup \{s_1\}$
- $\mathcal{Q}_i = \mathcal{Q}_{i-1} \cup \{s_i\} \quad i = 2, \ldots, h,$

and for what already proved

$$v(\gamma; \mathcal{P}) \leq v(\gamma; \mathcal{Q}_1) \leq v(\gamma; \mathcal{Q}_2) \leq \cdots \leq v(\gamma; \mathcal{Q}_h) = v(\gamma; \mathcal{Q}). \tag{4.3}$$

(b) For a partition $\mathcal{P} = \{t_0, \ldots, t_n\}$ we have

$$\begin{aligned} v(\alpha\gamma + \beta\sigma; \mathcal{P}) &= \sum_{k=1}^{j} |(\alpha\gamma + \beta\sigma)(t_k) - (\alpha\gamma + \beta\sigma)(t_{k-1})| = \\ &= \sum_{k=1}^{j} |\alpha\gamma(t_k) + \beta\sigma(t_k) - \alpha\gamma(t_{k-1}) - \beta\sigma(t_{k-1})| \leq \\ &\leq \sum_{k=1}^{j} |\alpha(\gamma(t_k) - \gamma(t_{k-1}))| + |\beta(\sigma(t_k) - \sigma(t_{k-1}))| = \\ &= \sum_{k=1}^{j} |\alpha|\, |\gamma(t_k) - \gamma(t_{k-1})| + |\beta|\, |\sigma(t_k) - \sigma(t_{k-1}))| = \\ &= |\alpha|\, v(\gamma; \mathcal{P}) + |\beta|\, v(\sigma; \mathcal{P}) \leq |\alpha|\, V(\gamma) + |\beta|\, V(\sigma) \end{aligned}$$

whence $V(\alpha\gamma + \beta\sigma) \leq |\alpha|\, V(\gamma) + |\beta|\, V(\sigma)$.

4.1.3 Exercise

4.1.4 Exercise

4.1.5 Exercise

We have, for $t \in (0, 1]$,

$$\gamma'(t) = \frac{e^{-\frac{1}{t}}}{t^2} \left[\cos\frac{1}{t} + \sin\frac{1}{t} + i\left(\sin\frac{1}{t} - \cos\frac{1}{t}\right)\right] \tag{4.4}$$

and

$$|\gamma'(t)| = \sqrt{2}\, \frac{e^{-\frac{1}{t}}}{t^2}. \tag{4.5}$$

Then for $0 < h < 1$, the path

$$\begin{aligned} \gamma_h : [h, 1] &\to \mathbb{C} \\ t &\mapsto \gamma(t) \end{aligned}$$

is smooth, and

$$V(\gamma_h) = \int_h^1 \sqrt{2}\, \frac{e^{-\frac{1}{t}}}{t^2} dt = \sqrt{2}(e^{-1} - e^{-\frac{1}{h}}). \tag{4.6}$$

Hence by 9.1.2 γ is rectifiable, and

$$V(\gamma) = \lim_{h \to 0^+} \int_h^1 \sqrt{2}\, \frac{e^{-\frac{1}{t}}}{t^2} dt = \sqrt{2}(e^{-1} - e^{-\frac{1}{h}}) = \frac{\sqrt{2}}{e}. \tag{4.7}$$

4.1.6 Exercise

4.1.7 Exercise

Since

$$\lim_{t \to 0} \gamma(t) = 0 = \gamma(0) \tag{4.8}$$

γ is a path.

Now, let

$$\begin{cases} t'_k = \dfrac{2}{(1+4k)\pi} & k \in \mathbb{N} \\ t''_k = \dfrac{2}{(3+4k)\pi} & k \in \mathbb{N} \end{cases} \tag{4.9}$$

and

$$P_N = \left[0, t''_N, t'_N, t''_{N-1}, t'_{N-1}, \ldots, t''_0, t'_0, 1\right]. \tag{4.10}$$

Then

$$\begin{aligned} v(\gamma, P_N) = |\gamma(t''_N) - \gamma(0)| + \sum_{k=0}^{N} |\gamma(t'_k) - \gamma(t''_k)| + \\ + \sum_{k=1}^{N} \left|\gamma(t''_{k-1}) - \gamma(t'_k)\right| + |\gamma(1) - \gamma(t'_0)| \geq \\ \geq \sum_{k=0}^{N} \left|\gamma\left(\frac{2}{(1+4k)\pi}\right) - \gamma\left(\frac{2}{(3+4k)\pi}\right)\right| = \\ = \sum_{k=0}^{N} \left|\frac{2}{(1+4k)\pi}(1+i) - \frac{2}{(3+4k)\pi}(1-i)\right| = \\ = \sum_{k=0}^{N} \left|\frac{4}{(1+4k)(3+4k)\pi} + i\frac{8(1+2k)}{(1+4k)(3+4k)\pi}\right| \geq \\ \geq \sum_{k=0}^{N} \frac{8(1+2k)}{(1+4k)(3+4k)\pi} \end{aligned}$$

and the last series diverges.

4.1.8 Exercise

4.1.9 Exercise

We have

$$\int_\gamma \frac{1}{z}\ dz = \int_0^{2\pi} \frac{in}{e^{int}} e^{int} dt = 2\pi in.$$

4.1.10 Exercise

We have

$$\int_\gamma z^n\ dz = i \int_0^{2\pi} e^{int} e^{it} dt = i \int_0^{2\pi} e^{i(n+1)t} dt.$$

If $n \neq -1$

$$\int_\gamma z^n\ dz = i \left[\frac{e^{i(n+1)t}}{(n+1)t}\right]_0^{2\pi} = 0.$$

If $n = -1$

$$\int_\gamma z^n\ dz = i\,[t]_0^{2\pi} = 2\pi i.$$

4.1.11 Exercise

Let

$$\begin{aligned}\gamma_1(t) &= 2it + 1 - i\\ \gamma_2(t) &= -2t + 1 + i\\ \gamma_3(t) &= -2it - 1 + i\\ \gamma_4(t) &= 2t - 1 - i.\end{aligned}$$

Then

$$\int_\gamma \frac{1}{z}\,dz = \sum_{j=1}^{4}\int_{\gamma_j}\frac{1}{z}\,dz.$$

If log is any branch of the logarithm defined on an open subset of $\mathbb{C}$ containing the path on which each integral is to be calculated, we have

$$\begin{aligned}\int_{\gamma_1}\frac{1}{z}\,dz &= \int_0^1 \frac{2i}{2it+1-i}\,dt = [\log(2it+1-i)]_0^1 = \frac{\pi}{2}i\\ \int_{\gamma_2}\frac{1}{z}\,dz &= \int_0^1 \frac{-2}{-2t+1+i}\,dt = [\log(-2t+1+i)]_0^1 = \frac{\pi}{2}i\\ \int_{\gamma_3}\frac{1}{z}\,dz &= \int_0^1 \frac{-2i}{-2it-1+i}\,dt = [\log(-2it-1+i)]_0^1 = \frac{\pi}{2}i\\ \int_{\gamma_4}\frac{1}{z}\,dz &= \int_0^1 \frac{2}{2t-1-i}\,dt = [\log(2t-1-i)]_0^1 = \frac{\pi}{2}i\end{aligned}$$

so

$$\int_\gamma \frac{1}{z}\,dz = 2\pi i.$$

4.1.12 Exercise

We have

$$\begin{aligned}I(r) &= \int_0^\pi \frac{e^{ire^{it}}}{re^{it}} ire^{it}dt = i\int_0^\pi e^{ir(\cos t + i\sin t)}dt =\\ &= i\int_0^\pi e^{-r\sin t}e^{ir\cos t}dt;\end{aligned}$$

then

$$\begin{aligned}|I(r)| &\le \int_0^\pi \left|e^{-r\sin t}e^{ir\cos t}\right|dt = \int_0^\pi \left|e^{-r\sin t}\right|\left|e^{ir\cos t}\right|dt =\\ &= \int_0^\pi e^{-r\sin t}dt = \pi(e^{-\sin\bar{t}})^r\end{aligned}$$

for some $\bar{t} \in (0,\pi)$, whence $e^{-\sin\bar{t}} < 1$, and

$$\lim_{r\to+\infty}|I(r)| = 0. \tag{4.11}$$

4.1.13 Exercise

What is $z^{-\frac{1}{2}}$ supposed to mean? There is no way to define a branch of the logarithm on an open subset of $\mathbb{C}$ in which both paths are contained. Let

$$\begin{aligned}&D_1\left\{z\in\mathbb{C} \mid \mathfrak{Im}(z)\in(-\frac{\pi}{2},\frac{3\pi}{2})\right\}\\ &D_2\left\{z\in\mathbb{C} \mid \mathfrak{Im}(z)\in(\frac{\pi}{2},\frac{5\pi}{2})\right\}.\end{aligned}$$

(a) If $\log' = \exp^{-1}_{|D_1}$ and $z^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\log'(z)\right)$

$$\int_\gamma z^{-\frac{1}{2}} dz = i\int_0^\pi e^{\frac{1}{2}it} dt = 2\left[e^{\frac{1}{2}it}\right]_0^\pi = 2\,[i-1]\,.$$

(b) If $\log'' = \exp^{-1}_{|D_2}$ and $z^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\log''(z)\right)$

$$\int_\gamma z^{-\frac{1}{2}} dz = i\int_0^\pi e^{-\frac{1}{2}it} dt = -2\left[e^{-\frac{1}{2}it}\right]_0^\pi = 2\,[i+1]\,.$$

But let

$$D_3\left\{z\in\mathbb{C}\mid \mathfrak{Im}(z)\in(\frac{3\pi}{2},\frac{7\pi}{2})\right\}$$
$$D_4\left\{z\in\mathbb{C}\mid \mathfrak{Im}(z)\in(\frac{5\pi}{2},\frac{9\pi}{2})\right\}.$$

(a) If $\log' = \exp^{-1}_{|D_3}$ and $z^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\log'(z)\right)$

$$\int_\gamma z^{-\frac{1}{2}} dz = i\int_0^\pi e^{\frac{1}{2}it+\pi i} dt = -2\left[e^{\frac{1}{2}it}\right]_0^\pi = -2\,[i-1]\,.$$

(b) If $\log'' = \exp^{-1}_{|D_4}$ and $z^{-\frac{1}{2}} = \exp\left(-\frac{1}{2}\log''(z)\right)$

$$\int_\gamma z^{-\frac{1}{2}} dz = i\int_0^\pi e^{-\frac{1}{2}it+\pi i} dt = 2\left[e^{-\frac{1}{2}it}\right]_0^\pi = -2\,[i+1]\,.$$

4.1.14 Exercise

4.1.15 Exercise

4.1.16 Exercise

4.1.17 Exercise

4.1.18 Exercise

4.1.19 Exercise

We have

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{1}{2}\left[\int_\gamma \frac{1}{z-1}\,dz - \int_\gamma \frac{1}{z+1}\,dz\right].$$

The second integral yelds 0 because for $t\in[0,2\pi]$ the point $\gamma(t)+1 = e^{it}+2$ lies in the domain of a branch of the logarithm. Then

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{i}{2}\int_0^{2\pi} \frac{e^{it}}{1+e^{it}-1} dt = \pi i.$$

4.1.20 Exercise

We have

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{1}{2}\left[\int_\gamma \frac{1}{z-1}\,dz - \int_\gamma \frac{1}{z+1}\,dz\right].$$

Now

$$\int_\gamma \frac{1}{z-1}\,dz = \int_{-\pi}^{\pi} \frac{2ie^{it}}{2e^{it}-1}\,dt = \int_{-\pi}^{0} \frac{2ie^{it}}{2e^{it}-1}\,dt + \int_0^{\pi} \frac{2ie^{it}}{2e^{it}-1}\,dt$$

$$\int_\gamma \frac{1}{z+1}\,dz = \int_{-\pi}^{\pi} \frac{2ie^{it}}{2e^{it}+1}\,dt = \int_{-\pi}^{0} \frac{2ie^{it}}{2e^{it}+1}\,dt + \int_{0}^{\pi} \frac{2ie^{it}}{2e^{it}+1}\,dt.$$

Letting

$$D_1 \left\{ z \in \mathbb{C} \mid \mathfrak{Im}(z) \in (-\frac{3\pi}{2}, \frac{\pi}{2}) \right\}$$
$$D_2 \left\{ z \in \mathbb{C} \mid \mathfrak{Im}(z) \in (-\frac{\pi}{2}, \frac{3\pi}{2}) \right\}$$

and $\log' = \exp^{-1}_{|D_1}$, $\log'' = \exp^{-1}_{|D_2}$, we have

$$\int_{-\pi}^{0} \frac{2ie^{it}}{2e^{it}-1}\,dt = \left[\log'(2e^{it}-1)\right]_{-\pi}^{0} = -\log'(-3) = -\log 3 + \pi i$$

$$\int_{0}^{\pi} \frac{2ie^{it}}{2e^{it}-1}\,dt = \left[\log''(2e^{it}-1)\right]_{0}^{\pi} = \log''(-3) = \log 3 + \pi i$$

$$\int_{-\pi}^{0} \frac{2ie^{it}}{2e^{it}+1}\,dt = \left[\log'(2e^{it}+1)\right]_{-\pi}^{0} = \log'(3) - \log'(-1) = \log 3 + \pi i$$

$$\int_{0}^{\pi} \frac{2ie^{it}}{2e^{it}+1}\,dt = \left[\log''(2e^{it}+1)\right]_{0}^{\pi} = \log''(-1) - \log''(3) = \pi i - \log 3.$$

Finally

$$\int_\gamma \frac{1}{z^2-1}\,dz = \frac{1}{2}\left[2\pi i + 2\pi i\right] = 2\pi i.$$

4.1.21 Exercise

Simply $(F_1 - F_2)' = 0$ implies $F_1 - F_2$ constant in G. We know that $F' = 0$ implies that F is constant also when F is only differentiable.

4.1.22 Exercise

For $n \geq 2$ let

$$f_n(z) = (z-a)^{-n}.$$

Then

$$f_n = \frac{f'_{n-1}}{1-n}$$

so each f_n has a primitive in G.

4.1.23 Exercise

We have

$$(fg)' = f'g + fg'$$

whence

$$\int_\gamma (fg)'dz = \int_\gamma f'g dz + \int_\gamma fg' dz$$

and

$$f(b)g(b) - f(a)g(a) = \int_\gamma f'g dz + \int_\gamma fg' dz.$$

4.2 Power series representation of analytic functions

4.2.1 Exercise

4.2.2 Exercise

4.2.3 Exercise

4.2.4 Exercise

4.2.5 Exercise

Actually Abel's Limit Theorem has a more general statement: see Ahlfors "Complex Analysis" 2.2.5 p. 42. To prove the given statement, let

$$S_n = \sum_{k=0}^{n} a_k$$

and for $|z| < 1$

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Then clearly

$$\lim_{n \to +\infty} S_n = A$$

and

$$\sum_{k=0}^{n} a_k x^k = S_0 + \sum_{k=1}^{n} (S_k - S_{k-1}) x^k = S_0 + \sum_{k=1}^{n} S_k x^k - \sum_{k=1}^{n} S_{k-1} x^k =$$
$$= \sum_{k=0}^{n-1} S_k x^k - \sum_{k=0}^{n-1} S_k x^{k+1} + S_n x^n = (1-x) \sum_{k=0}^{n-1} S_k x^k + S_n x^n$$

whence, for $|x| < 1$

$$f(x) = (1-x) \sum_{k=0}^{+\infty} S_k x^k.$$

Now for any $\epsilon > 0$ there is $m \in \mathbb{N}$ such that $n \geq m$ implies $|S_n - A| < \epsilon$. Then, for $|x| < 1$

$$\begin{aligned}
|f(x) - A| &= \left|(1-x)\sum_{k=0}^{+\infty} S_k x^k - A\right| = \\
&= \left|(1-x)\sum_{k=0}^{+\infty} S_k x^k - A(1-x)\sum_{k=0}^{+\infty} x^k\right| = \\
&= \left|(1-x)\sum_{k=0}^{+\infty}(S_k - A)x^k\right| = \\
&= \left|(1-x)\left(\sum_{k=0}^{m-1}(S_k - A)x^k + \sum_{k=m}^{+\infty}(S_k - A)x^k\right)\right| \leq \\
&\leq |1-x|\left(\left|\sum_{k=0}^{m-1}(S_k - A)x^k\right| + \left|\sum_{k=m}^{+\infty}(S_k - A)x^k\right|\right) = \\
&= |1-x|\left(\left|\sum_{k=0}^{m-1}(S_k - A)x^k\right| + |x|^m\left|\sum_{k=m}^{+\infty}(S_k - A)x^{k-m}\right|\right) = \\
&= |1-x|\left(\left|\sum_{k=0}^{m-1}(S_k - A)x^k\right| + |x|^m\left|\sum_{k=0}^{+\infty}(S_k - A)x^k\right|\right) \leq \\
&\leq |1-x|\left(\left|\sum_{k=0}^{m-1}(S_k - A)x^k\right| + |x|^m\left|\frac{1}{1-x}\right|\epsilon\right) \leq \\
&\leq |1-x|\left|\sum_{k=0}^{m-1}(S_k - A)x^k\right| + \epsilon
\end{aligned}$$

whence for any $\epsilon > 0$

$$\lim_{x\to 0^-} |f(x) - A| \leq \epsilon$$

that is

$$\lim_{x\to 0^-} |f(x) - A| = 0.$$

4.2.6 Exercise

4.2.7 Exercise

(a) Using integration by parts

$$\int_\gamma \frac{e^{iz}}{z^2}\,dz = \frac{1}{2}\int_\gamma \frac{ie^{iz}}{z^{-1}}\,dz = \frac{2\pi i}{2}\left[ie^{iz}\right]_{z=0} = \pi i.$$

(b)

$$\int_\gamma \frac{1}{z-a}\,dz = 2\pi i\,[1]_{z=a} = 2\pi i.$$

(c) Using integration by parts twice

$$\int_\gamma \frac{\sin z}{z^3}\,dz = \frac{1}{2}\int_\gamma \frac{\cos z}{z^2}\,dz = -\frac{1}{6}\int_\gamma \frac{\sin z}{z}\,dz = -\frac{1}{6}2\pi i\,[\sin z]_{z=0} = 0.$$

(d) Supposing log is the principal branch of the logartithm, since the integrand function is analytic in $B\left(1;\frac{1}{2}\right)$

$$\int_\gamma \frac{\log z}{z^n}\,dz = 0.$$

4.2.8 Exercise

4.2.9 Exercise

(a) Using integration by parts $n-1$ times

$$\int_\gamma \frac{2\sinh z}{z^n}\,dz = \frac{2}{n!}\begin{cases} \displaystyle\int_\gamma \frac{\sinh z}{z}\,dz & \text{if } n \text{ is odd} \\ \displaystyle\int_\gamma \frac{\cosh z}{z}\,dz & \text{if } n \text{ is even} \end{cases}$$

that is

$$\int_\gamma \frac{2\sinh z}{z^n}\,dz = \frac{2}{n!}2\pi i\begin{cases} [\sinh z]_{z=0} = 0 & \text{if } n \text{ is odd} \\ [\cosh z]_{z=0} = \dfrac{4\pi i}{n!} & \text{if } n \text{ is even} \end{cases}$$

(b) If $n > 1$ the integrand function has a primitive in $\mathbb{C} - \left\{\frac{1}{2}\right\}$, so the integral yields 0. If $n = 1$

$$\int_\gamma \frac{1}{\left(z-\frac{1}{2}\right)}\,dz = 2\pi i\,[1]_{z=\frac{1}{2}} = 2\pi i.$$

(c)

$$\int_\gamma \frac{1}{z^2+1}\,dz = \frac{1}{2i}\left[\int_\gamma \frac{1}{z-i}\,dz - \int_\gamma \frac{1}{z+i}\,dz\right] = \pi[1-1] = 0.$$

(d)

$$\int_\gamma \frac{\sin z}{z}\,dz = 2\pi i\,[\sin z]_{z=0} = 0.$$

(e) Supposing

$$z^{\frac{1}{m}} = e^{\frac{1}{m}\log z}$$

where log is the principal branch of the logarithm, then, using imtegration by parts $n-1$ times

$$\begin{aligned}\int_\gamma \frac{z^{\frac{1}{m}}}{(z-1)^m}\,dz &= \frac{(1-m)(1-2m)\cdots(1-(m-2)m)}{m^{m-1}m!}\int_\gamma \frac{z^{\frac{1-(m-1)m}{m}}}{z-1}\,dz = \\ &= \frac{(1-m)(1-2m)\cdots(1-(m-2)m)}{m^{m-1}m!}2\pi i\left[z^{\frac{1-(m-1)m}{m}}\right]_{z=1} = \\ &= \frac{(1-m)(1-2m)\cdots(1-(m-2)m)}{m^{m-1}m!}2\pi i.\end{aligned}$$

4.2.10 Exercise

We have

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz = \frac{1}{4}\int_\gamma \frac{1}{z}\,dz + \frac{3}{8}\int_\gamma \frac{1}{z-2i}\,dz + \frac{3}{8}\int_\gamma \frac{1}{z+2i}\,dz$$

then

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}\,dz = \begin{cases} \dfrac{\pi}{2}i & \text{if } r < 2 \\ 2\pi i & \text{if } r > 2 \end{cases}.$$

4.2.11 Exercise

If T is the Möbius transformation

$$T(z) = \frac{iz+1}{-iz+1}$$

we have

$$\begin{cases} T^{-1}(0) = i \\ T^{-1}(-1) = \infty \\ T^{-1}(\infty) = -i \end{cases}$$

so T maps the imaginary axis onto the real one, and the set $\{z \in \mathbb{C} \mid \mathfrak{Re}(z) = 0 \wedge |\mathfrak{Im}(z)| > 1\}$ onto the set $\{z \in \mathbb{C} \mid \mathfrak{Im}(z) = 0 \wedge \mathfrak{Re}(z) < 0\}$. So the domain of analyticity of f is

$$D = \mathbb{C} - \{z \in \mathbb{C} \mid \mathfrak{Re}(z) = 0 \wedge |\mathfrak{Im}(z)| > 1\}.$$

Furhtermore

$$\tan(z) = \frac{1}{i}\frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = \frac{1}{i}\frac{e^{2iz} - 1}{e^{2iz} + 1}$$

so

$$\tan(f(z)) = \frac{1}{i}\frac{\frac{iz+1}{-iz+1} - 1}{\frac{iz+1}{-iz+1} + 1} = \frac{1}{i}\frac{1+iz-1+iz}{1+iz+1-iz} = z.$$

If $|z| < i$

$$\begin{cases} \log(1+iz) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} i^k z^k \\ \log(1-iz) = \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k} (-i)^k z^k \end{cases}.$$

Now $\mathfrak{Re}(1+iz) > 0$ and $\mathfrak{Re}(1-iz) > 0$, whence $-\pi < \mathfrak{Im}(\log(1+iz) - \log(1-iz)) < \pi$ which means that $\log(1+iz) - \log(1-iz)$ lies in the image of log, and

$$\begin{aligned} \log(1+iz) - \log(1-iz) = \log\left(\frac{1+iz}{1-iz}\right) &= \\ &= \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k}(1-(-1)^k) i^k z^k = \\ &= \sum_{k=0}^{+\infty} \frac{2}{2k+1} i^{2k+1} z^{2k+1} = \\ &= \sum_{k=0}^{+\infty} \frac{2}{2k+1}(-1)^k i z^{2k+1}. \end{aligned}$$

Finally

$$f(z) = \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k+1} z^{2k+1}.$$

4.2.12 Exercise

Since $\sec(z) = \cos(z)^{-1}$, sec is an even function and its power series expantion has only even powers. Let

$$\sec(z) = \sum_{k=0}^{+\infty} \frac{E_{2k}}{(2k)!} z^{2k}.$$

Now $\cos(z)\sec(z) = 1$ yields $E_0 = 1$ and for $n \geq 1$

$$\sum_{k=0}^{n} \frac{(-1)^k}{(2k)!} \frac{E_{2(n-k)}}{(2(n-k))!} = 0$$

whence

$$\sum_{k=0}^{n} (-1)^k \frac{(2n)!}{(2k)!(2(n-k))!} E_{2(n-k)} = 0$$

that is

$$\sum_{k=0}^{n} (-1)^k \binom{2n}{2(n-k)} E_{2(n-k)} = 0.$$

Since the domain of sec is $D = \mathbb{C} - \left\{\frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\right\}$ the radius of convergence of the series is $d(0, D) = \frac{\pi}{2}$.

4.2.13 Exercise

To be more accurate, define

$$g(z) = \begin{cases} \dfrac{e^z - 1}{z} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}.$$

Then

$$g(z) = \sum_{k=0}^{+\infty} \frac{z^k}{(k+1)!}.$$

Since g is defined everywhere, the radius of convergence of the last series is $+\infty$. Alternatively

$$\lim_{k \to +\infty} \sqrt[k]{\left|\frac{1}{(k+1)!}\right|} = 0.$$

In the same way define

$$f(z) = \begin{cases} \dfrac{z}{e^z - 1} & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}.$$

Then $g(z)f(z) = 1$ for each $z \in \mathbb{C}$. Let

$$f(z) = \sum_{k=0}^{+\infty} \frac{a_k}{k!} z^k.$$

Since f is defined in $D = \mathbb{C} - \{2k\pi i, k \in \mathbb{Z}\}$, the last series has radius of convergence 2π. Since $g(z)f(z) = 1$

$$\sum_{k=0}^{n} \frac{a_k}{k!} \frac{1}{(n-k+1)!} = 0$$

or

$$\sum_{k=0}^{n} a_k \frac{(n+1)!}{k!(n-k+1)!} = \sum_{k=0}^{n} a_k \binom{n+1}{k} = 0.$$

Now

$$h(z) = f(z) + \frac{1}{2} z = \frac{1}{2} \frac{e^z + 1}{e^z - 1} z$$

and this is an even function, so

$$h(z) = \sum_{k=0}^{+\infty} \frac{a_k}{k!} z^k + \frac{1}{2} z = 1 + \left(a_1 + \frac{1}{2}\right) z + \sum_{k=2}^{+\infty} \frac{a_k}{k!} z^k$$

which yields

$$\begin{cases} a_1 = -\dfrac{1}{2} \\ a_{2k+1} = 0, \quad k > 0 \end{cases}.$$

4.2.14 Exercise

If

$$h(z) = \frac{1}{2} \frac{e^z + 1}{e^z - 1} z$$

we have

$$\cot(z) = i \frac{e^{2iz} + 1}{e^{2iz} - 1} = \frac{1}{z} \left(\frac{1}{2} \frac{e^{2iz} + 1}{e^{2iz} - 1} 2iz \right) = \frac{1}{z} h(2iz)$$

and

$$\begin{aligned}
\tan(z) = \cot(z) - 2\cot(2z) = \frac{1}{z} h(2iz) - \frac{1}{z} h(4iz) = \\
= \frac{1}{z} \left(\sum_{k=0}^{+\infty} \frac{a_{2k}}{(2k)!} (2iz)^{2k} - \sum_{k=0}^{+\infty} \frac{a_{2k}}{(2k)!} (4iz)^{2k} \right) = \\
= \frac{1}{z} \left(\sum_{k=0}^{+\infty} \frac{a_{2k}}{(2k)!} \left((2iz)^{2k} - (4iz)^{2k} \right) \right) = \\
= \frac{1}{z} \left(\sum_{k=1}^{+\infty} \frac{a_{2k}}{(2k)!} (2i)^{2k} \left(1 - 2^{2k}\right) z^{2k} \right) = \\
= \frac{1}{z} \left(\sum_{k=1}^{+\infty} \frac{a_{2k}}{(2k)!} (-1)^k 4^k \left(1 - 2^{2k}\right) z^{2k} \right) = \\
= \sum_{k=0}^{+\infty} \frac{a_{2(k+1)}}{(2(k+1))!} (-1)^{k+1} 4^{k+1} \left(1 - 2^{2(k+1)}\right) z^{2k+1} = \\
= \sum_{k=0}^{+\infty} \frac{B_{2(k+1)}}{(2(k+1))!} 4^{k+1} \left(1 - 2 \cdot 4^k\right) z^{2k+1}.
\end{aligned}$$

4.3 Zeroes of an Analythic function

4.3.1 Exercise

Let

$$f(z) = \sum_{k=0}^{+\infty} a_k z^k.$$

Since

$$\frac{f(z)}{z^n} = \sum_{k=0}^{n-1} a_k z^{k-n} + \sum_{k=n}^{+\infty} a_k z^{k-n}$$

then if $|z| > R$

$$\left| \sum_{k=n}^{+\infty} a_k z^k \right| < \left| \frac{f(z)}{z^n} \right| + \sum_{k=0}^{n-1} |a_k| \frac{1}{R^{n-k}} \leq M + \sum_{k=0}^{n-1} |a_k| \frac{1}{R^{n-k}}.$$

This implies that

$$\sum_{k=0}^{+\infty} a_{n+k} z^k$$

is a constant, that is, $a_{n+k} = 0$ if $k > 0$, and

$$f(z) = \sum_{k=0}^{n} a_k z^k.$$

4.3.2 Exercise

Let

$$G_1 = \{z \in \mathbb{C} \mid |z+2| < 1\}$$

$$G_2 = \{z \in \mathbb{C} \mid |z-2| < 1\}$$

$G = G_1 \cup G_2$ and $f : G \to \mathbb{C}$ defined by

$$f(z) = \begin{cases} 0 & \text{if } z \in G_1 \\ 1 & \text{if } z \in G_2 \end{cases}.$$

4.3.3 Exercise

Let f, g be entire functions such that $f(x) = e^x$ and $g(x) = e^x$ if $x \in \mathbb{R}$. Then the set

$$\{z \in \mathbb{C} \mid f(z) = g(z)\}$$

has a limit point in $\mathbb{C}$ so $f = g$.

4.3.4 Exercise

4.3.5 Exercise

4.3.6 Exercise

4.3.7 Exercise

4.3.8 Exercise

4.3.9 Exercise

4.3.10 Exercise

The function

$$\frac{\overline{f}g}{g}$$

is analytic where $g \neq 0$, and so is $\overline{f}$, which is impossible.

4.4 The index of a closed curve

4.4.1 Exercise

4.4.2 Exercise

For each $k \in \mathbb{N}^+$ define the map

$$\alpha_k^1 : \left[\frac{1}{k+1}, \frac{1}{k}\right] \to [-\pi, \pi]$$
$$t \mapsto \pi[2k(k+1)t - 2k - 1]$$

and the path with the same domain

$$\gamma_k^1(t) = \frac{1}{4^k}\left(e^{i\alpha_k^1(t)} + 1\right).$$

Clearly

$$V(\gamma_k^1) = \frac{\pi}{2^{k-1}}$$
$$\gamma_k^1\left(\frac{1}{k}\right) = 0$$
$$\gamma_k^1\left(\frac{1}{k+1}\right) = 0$$

so the path $\gamma^1 : [0,1] \to \mathbb{C}$ defined by

$$\gamma^1(t) = \begin{cases} \gamma_k^1(t) & \text{if } t \in \left(\frac{1}{k+1}, \frac{1}{k}\right] \quad k \in \mathbb{N} \\ 0 & \text{if } t = 0 \end{cases}$$

is continuous, and

$$\sum_{k=0}^{+\infty} V(\gamma_k^1)$$

converges. By Proposition 9.1.3 γ^1 is a rectifiable path.

In the same way for each $k \in \mathbb{N}^+$ define the map

$$\alpha_k^2 : \left[-\frac{1}{k}, -\frac{1}{k+1}\right] \to [0, 2\pi]$$
$$t \mapsto -\pi[2k(k+1)t + 2k]$$

and the path with the same domain

$$\gamma_k^2(t) = \frac{1}{4^k}\left(e^{i\alpha_k^2(t)} - 1\right)$$

so the path $\gamma^2 : [-1,0] \to \mathbb{C}$ defined by

$$\gamma^2(t) = \begin{cases} \gamma_k^2(t) & \text{if } t \in \left[-\frac{1}{k}, -\frac{1}{k+1}\right) \quad k \in \mathbb{N} \\ 0 & \text{if } t = 0 \end{cases}$$

is continuous and rectifiable. Let $\gamma : [-1,1] \to \mathbb{C}$ defined by

$$\gamma(t) = \begin{cases} \gamma^1(t) & \text{if } t \in [0,1] \\ \gamma^2(t) & \text{if } t \in [-1,0]. \end{cases}$$

Let

$$p_h^1 = \left(\frac{1}{4^h}, 0\right) \quad h \in \mathbb{N}$$
$$p_h^2 = \left(-\frac{1}{4^h}, 0\right) \quad h \in \mathbb{N}.$$

Again by Proposition 9.1.3

$$n(\gamma; p_h^1) = \frac{1}{2\pi i}\int_\gamma \frac{1}{p_h^1 - z}\,dz = \sum_{k=1}^{+\infty}\int_{\gamma_k^1} \frac{1}{p_h^1 - z}\,dz + \sum_{k=1}^{+\infty}\int_{\gamma_k^2} \frac{1}{p_h^1 - z}\,dz.$$

that is

$$n(\gamma; p_h^1) = \sum_{k=1}^{+\infty} n(\gamma_k^1; p_h^1) + \sum_{k=1}^{+\infty} n(\gamma_k^2; p_h^1).$$

Since clearly p_h^1 belongs to the unboundeded component of $\mathbb{C} - \left\{\gamma_k^2\right\}$ for all k, to the unbounded component of $\mathbb{C} - \left\{\gamma_k^1\right\}$ if $k > h$ and to the only one bound component of $\mathbb{C} - \left\{\gamma_k^1\right\}$ if $k \leq h$, then $n(\gamma; p_k^1) = k$. Similarly $n(\gamma; p_k^2) = -k$.

4.4.3 Exercise

Let

$$p(z) = a \prod_{k=1}^{n} (z - z_k)$$

where $|z_k| < R,\ k = 1 \dots n$. Then

$$p'(z) = a \sum_{k=1}^{n} \prod_{h \neq k} (z - z_h)$$

so

$$\frac{p'(z)}{p(z)} = \sum_{k=1}^{n} \frac{1}{z - z_k}$$

and

$$\int_\gamma \frac{p'(z)}{p(z)}\, dz = \int_\gamma \sum_{k=1}^{n} \frac{1}{z - z_k}\, dz = 2\pi i n.$$

4.4.4 Exercise

Let

$$\begin{aligned} \sigma_1 :\ & [0,1] \to \mathbb{C} \\ & t \mapsto (1 + (r-1)t)e^{i\theta} \end{aligned}$$

and

$$\begin{aligned} \sigma_2 :\ & [0,1] \to \mathbb{C} \\ & t \mapsto e^{it\theta} \end{aligned}$$

Then there is an integer k such that

$$\int_\gamma \frac{1}{z}\, dz + \int_{-\sigma_1} \frac{1}{z}\, dz + \int_{-\sigma_2} \frac{1}{z}\, dz = 2k\pi i$$

whence

$$\begin{aligned} \int_\gamma \frac{1}{z}\, dz &= \int_0^1 \frac{r-1}{1+(r-1)t}\, dt + i \int_0^1 \theta dt + 2k\pi i = \\ &= [\log(1 + (r-1)t)]_0^1 + [i\theta]_0^1 + 2k\pi i = \log r + i\theta + 2k\pi i. \end{aligned}$$

4.5 Cauchy's Theorem and Integral Formula

4.5.1 Exercise

In every point $(v, w) \in G \times G$ such that $v \neq w$ the function φ is clearly continuous.
Let r be such that $\overline{B}(w, r) \subseteq G$. If $u, v \in B(w, r/4)$ then $u \in B(v, r/2)$ and $B(v, r/2) \subseteq B(w, r)$. So f is analytic in $B(v, r/2)$ and

$$f(u) - f(v) = (u - v) \sum_{k=1}^{+\infty} \frac{f^{(k)}(v)}{k!} (u - v)^{k-1} = (u - v)f^{'}(v) + (u - v)^2 \sum_{k=2}^{+\infty} \frac{f^{(k)}(v)}{k!} (u - v)^{k-2}$$

whence if $u \neq v$ and $(u, v) \neq (w, w)$

$$\begin{aligned} \left| \frac{f(u) - f(v)}{u - v} - f^{'}(w) \right| &= \left| f^{'}(v) + (u - v) \sum_{k=2}^{+\infty} \frac{f^{(k)}(v)}{k!} (u - v)^{k-2} - f^{'}(w) \right| \\ &\leq \left| f^{'}(v) - f^{'}(w) \right| + |u - v| \left| \sum_{k=2}^{+\infty} \frac{f^{(k)}(v)}{k!} (u - v)^{k-2} \right|. \end{aligned}$$

Now if $|f(z)| \leq M$ in $B(w,r)$ also $|f(z)| \leq M$ holds in $B(v,r/2)$ for any v in $B(w,r/4)$, then by Cauchy's Estimate

$$\left|f^{(k)}(v)\right| \leq \frac{k!M2^k}{r^k}$$

for any v in $B(w,r/4)$. It follows that

$$\left|\sum_{k=2}^{+\infty} \frac{f^{(k)}(v)}{k!}(u-v)^{k-2}\right| \leq M\sum_{k=2}^{+\infty}\frac{2^k|u-v|^{k-2}}{r^k} = \frac{4M}{r^2}\sum_{k=0}^{+\infty}\left(\frac{2|u-v|}{r}\right)^k$$

and, since $|u-v| < r/2$

$$\left|\sum_{k=2}^{+\infty} \frac{f^{(k)}(v)}{k!}(u-v)^{k-2}\right| \leq \frac{4M(r-2|u-v|)}{r}.$$

Eventually

$$\left|\frac{f(u)-f(v)}{u-v} - f^{'}(w)\right| \leq \left|f^{'}(v)-f^{'}(w)\right| + |u-v|\frac{4M(r-2|u-v|)}{r}.$$

So if $\epsilon > 0$ there is δ_1 such that for any (u,v) such that $u \neq v$, $(u,v) \neq (w,w)$ and $\sqrt{(u-w)^2+(v-w)^2} < \delta_1$ the inequality $|\varphi(u,v)-\varphi(w,w)| < \epsilon$ holds.
Since $f^{'}$ is a continuous function there is δ_2 such that for any (u,u) such that $(u,u) \neq (w,w)$ and $\sqrt{(u-w)^2+(u-w)^2} = \sqrt{2}|u-w| < \delta_2$ the inequality $|\varphi(u,u)-\varphi(w,w)| = \left|f^{'}(u)-f^{'}(v)\right| < \epsilon$ holds.
If $\delta = \min\{\delta_1,\delta_2\}$ then $(u,v) \neq (w,w)$ and $\sqrt{(u-w)^2+(v-w)^2} < \delta$ implies $|\varphi(u,v)-\varphi(w,w)| < \epsilon$.

For $z \in G$ let

$$\varphi_v(z) = \varphi(z,v).$$

Clearly φ_v is analytic for any $u \in G$ such that $u \neq v$. Let r be such that $B(v,r) \subseteq G$. For $z \neq v$ and $|z-v| < r$

$$\frac{\varphi_v(z)-\varphi_v(z)}{z-v} = \frac{1}{z-v}\left(\frac{f(z)-f(v)}{z-v} - f^{'}(v)\right) = \frac{1}{z-v}\left(\sum_{k=1}^{+\infty}\frac{f^{(k)}(v)}{k!}(z-v)^{k-1} - f^{'}(v)\right) =$$
$$= \frac{1}{z-v}\left(\sum_{k=2}^{+\infty}\frac{f^{(k)}(v)}{k!}(z-v)^{k-1}\right) = \sum_{k=2}^{+\infty}\frac{f^{(k)}(v)}{k!}(z-v)^{k-2}$$

whence

$$\lim_{z\to v}\frac{\varphi_v(z)-\varphi_v(z)}{z-v} = \frac{f^{''}(v)}{2}.$$

If $z \neq v$

$$\varphi^{'}(z) = \frac{f^{'}(z)(z-v)-(f(z)-f(v))}{(z-v)^2} =$$
$$= \frac{f^{'}(z)(z-v)-f^{'}(v)(z-v)-\frac{f^{''}(v)}{2}(z-v)^2-\sum_{k=3}^{+\infty}\frac{f^{(k)}(v)}{k!}(z-v)^k}{(z-v)^2} =$$
$$= \frac{f^{'}(z)-f^{'}(v)}{z-v} - \frac{f^{''}(v)}{2} - \sum_{k=3}^{+\infty}\frac{f^{(k)}(v)}{k!}(z-v)^{k-2}$$

so

$$\lim_{z\to v}\varphi_v^{'}(z) = \frac{f^{''}(v)}{2} = \varphi^{'}(v).$$

NB Remember we still don't know that a derivable function is analytic.

4.5.2 Exercise

4.5.3 Exercise

Chapter 5

Singularities

5.1 Classification of singularities

5.2 Residues r_2 r_1

5.2.1 Exercise

(h,0)

(a)

The polynomial $z^4 + z^2 + 1$ has roots $r_1 = e^{i\frac{\pi}{3}}, r_2 = e^{i\frac{2\pi}{3}}, r_3 = e^{i\frac{4\pi}{3}}, r_4 = e^{i\frac{5\pi}{3}}$. The residues of the function

$$f(z) = \frac{z^2}{z^4 + z^2 + 1}$$

are

$$\begin{aligned} \operatorname{res}(f, r_1) &= \frac{\sqrt{3} - i}{4\sqrt{3}} \\ \operatorname{res}(f, r_2) &= \frac{-\sqrt{3} - i}{4\sqrt{3}}. \end{aligned}$$

Let $h > 1$ and

$$\begin{aligned} \gamma_h :\ & [0, \pi] \to \mathbb{C} \\ & t \mapsto he^{it} \\ \sigma_h :\ & [-h, h] \to \mathbb{C} \\ & t \mapsto t \end{aligned}.$$

Then

$$\int_{\gamma_h + \sigma_h} f(z)\, dz = 2\pi i \left(\operatorname{res}(f, r_1) + \operatorname{res}(f, r_2)\right) = \frac{\pi\sqrt{3}}{3}.$$

For the integral on γ_h we have

$$\left| \int_{\gamma_h} \frac{z^2}{z^4 + z^2 + 1}\, dz \right| = \left| i \int_0^\pi \frac{h^3 e^{3it}}{h^4 e^{4it} + h^2 e^{2it} + 1}\, dt \right| \le \int_0^\pi \frac{h^3}{|h^4 e^{4it} + h^2 e^{2it} + 1|}\, dt.$$

Now

$$\left| h^4 e^{4it} + h^2 e^{2it} + 1 \right| = \left| h^2 e^{2it}(h^2 e^{2it} + 1) + 1 \right| \ge \left| h^2 \left| h^2 e^{2it} + 1 \right| - 1 \right|$$

in turn, since $h > 1$

$$\left| h^2 e^{2it} + 1 \right| \ge h^2 - 1$$

and if $h > \sqrt{2}$

$$h^2 \left| h^2 e^{2it} + 1 \right| \ge 1$$

so

$$\left|h^2\left|h^2e^{2it}+1\right|-1\right| = h^2\left|h^2e^{2it}+1\right|-1 \geq h^2(h^2-1)-1.$$

Going back to the integral we have

$$\left|\int_{\gamma_h}\frac{z^2}{z^4+z^2+1}\,dz\right| \leq \int_0^\pi \frac{h^3}{h^2(h^2-1)-1}\,dt = \frac{h^3\pi}{h^2(h^2-1)-1}$$

that is

$$\lim_{h\to+\infty}\int_{\gamma_h}\frac{z^2}{z^4+z^2+1}\,dz = 0.$$

This yields

$$\lim_{h\to+\infty}\int_{\sigma_h}\frac{z^2}{z^4+z^2+1}\,dz = \frac{\pi\sqrt{3}}{3}.$$

But

$$\int_{\sigma_h}\frac{z^2}{z^4+z^2+1}\,dz = \int_{-h}^{h}\frac{t^2}{t^4+t^2+1}\,dt = 2\int_0^h\frac{t^2}{t^4+t^2+1}\,dt$$

which finally yields

$$\int_0^{+\infty}\frac{t^2}{t^4+t^2+1}\,dt = \frac{\pi\sqrt{3}}{6}.$$

(b) To calculate this integral we don't need the residues. Provided we already know that

$$\int_0^{+\infty}\frac{\sin x}{x} = \frac{\pi}{2}.$$

In fact, for $0 < a < b$

$$\int_a^b \frac{\cos x - 1}{x^2}\,dx = \left[-\frac{\cos x - 1}{x}\right]_a^b - \int_a^b\frac{\sin x}{x}\,dx$$

so

$$\int_0^\pi \frac{\cos x - 1}{x^2}\,dx = \left[-\frac{\cos x - 1}{x}\right]_0^\pi - \int_0^\pi\frac{\sin x}{x}\,dx = \frac{2}{\pi} - \int_0^\pi\frac{\sin x}{x}\,dx,$$

$$\int_\pi^{+\infty} \frac{\cos x - 1}{x^2}\,dx = \left[-\frac{\cos x - 1}{x}\right]_\pi^{+\infty} - \int_\pi^{+\infty}\frac{\sin x}{x}\,dx = -\frac{2}{\pi} - \int_\pi^{+\infty}\frac{\sin x}{x}\,dx,$$

and

$$\int_0^{+\infty} \frac{\cos x - 1}{x^2}\,dx = -\int_0^{+\infty}\frac{\sin x}{x}\,dx = -\frac{\pi}{2}.$$

(c) If $a = 0$ then

$$\int_0^\pi \frac{\cos 2\theta}{1-2a\cos\theta+a^2}\,d\theta = \int_0^\pi \cos 2\theta\,d\theta = 0$$

so suppose $a \neq 0$.
If $z = e^{i\theta}$ then $z^{-1} = e^{-i\theta}$, so

$$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

and

$$\cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right).$$

Hence, if $|z| = 1$

$$\frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} = -\frac{1}{2}\frac{z^4+1}{z(az^2 - (1+a^2)z + a)}$$

so if

$$\begin{aligned}\gamma : [0,1] &\to \mathbb{C}\\ t &\mapsto e^{it}\end{aligned}$$

we have

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta = -\frac{1}{4i}\int_\gamma \frac{z^4+1}{z^2(az^2 - (1+a^2)z + a)}\,dz.$$

The integrand function f has simple poles in a and $1/a$ and a double pole in 0. If $a^2 < 1$ only 0 and a lie in the interior of γ, and

$$\begin{aligned}\operatorname{res}(f,0) &= \frac{1+a^2}{a^2}\\ \operatorname{res}(f,a) &= \frac{1+a^4}{a^2(a^2-1)}.\end{aligned}$$

Eventually

$$\int_0^{\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta = -\frac{\pi}{2}\left(\operatorname{res}(f,0) + \operatorname{res}(f,a)\right) = \frac{\pi a^2}{1-a^2}.$$

(d) If $|z| = 1$ we have

$$\frac{1}{(a+\cos\theta)^2} = \frac{4z^2}{(2az + z^2 + 1)}$$

so if

$$\begin{aligned}\gamma : [0,\pi] &\to \mathbb{C}\\ t &\mapsto e^{it}\end{aligned}$$

we have

$$\int_0^{\pi} \frac{1}{(a+\cos\theta)^2} = \frac{1}{2i}\int_\gamma \frac{4z}{(2az + z^2 + 1)}.$$

The integrand function f has double poles in $-a \pm \sqrt{a^2-1}$, but only $-a + \sqrt{a^2-1}$ lies in the interior of γ, and

$$\operatorname{res}(f, -a + \sqrt{a^2-1}) = \frac{a}{\sqrt{a^2-1}^3}$$

so

$$\int_0^{\pi} \frac{1}{(a+\cos\theta)^2} = \frac{\pi a}{\sqrt{a^2-1}^3}.$$

5.2.2 Exercise

(a) Let

$$\begin{aligned}\gamma_h : [0,\pi] &\to \mathbb{C}\\ t &\mapsto he^{it}\end{aligned}$$

and

$$\begin{aligned}\delta_h : [-h,h] &\to \mathbb{C}\\ t &\mapsto t\end{aligned}.$$

The function

$$f(z) = \frac{1}{(z^2 + a^2)^2}$$

has double poles in $\pm ia$, and

$$\operatorname{res}(f, ia) = \frac{1}{4ia^3}.$$

Then if $h > a$

$$2\int_0^h \frac{1}{(t^2 + a^2)^2}\,dt + i\int_0^\pi \frac{he^{it}}{(h^2 e^{2it} + a^2)^2}\,dt = \int_{\gamma_h + \delta_h} \frac{1}{(z^2 + a^2)^2}\,dz = \frac{\pi}{2a^3}.$$

But

$$\left| \frac{he^{it}}{(h^2 e^{2it} + a^2)^2} \right| = \frac{h}{|h^2 e^{2it} + a^2|^2} \le \frac{h}{||h^2 e^{2it}| - a^2|^2} = \frac{h}{(h^2 - a^2)^2}$$

so

$$\lim_{h \to +\infty} \int_0^\pi \frac{he^{it}}{(h^2 e^{2it} + a^2)^2}\,dt = 0$$

and

$$\lim_{h \to +\infty} \int_0^h \frac{1}{(t^2 + a^2)^2}\,dt = \frac{\pi}{4a^3}.$$

(b) Let $h > 1$. Then

$$\begin{aligned}
\int_{1/h}^h \frac{(\log x)^3}{1 + x^2}\,dx &= \int_{1/h}^1 \frac{(\log x)^3}{1 + x^2}\,dx + \int_1^h \frac{(\log x)^3}{1 + x^2}\,dx = \\
&= \int_h^1 \frac{\left(\log \frac{1}{x}\right)^3}{1 + \left(\frac{1}{x}\right)^2} \left(-\frac{1}{x^2}\right)\,dx + \int_1^h \frac{(\log x)^3}{1 + x^2}\,dx = \\
&= -\int_h^1 \frac{(-\log x)^3}{1 + x^2}\,dx + \int_1^h \frac{(\log x)^3}{1 + x^2}\,dx = \\
&= -\int_1^h \frac{(\log x)^3}{1 + x^2}\,dx + \int_1^h \frac{(\log x)^3}{1 + x^2}\,dx = 0.
\end{aligned}$$

This imply

$$\int_0^{+\infty} \frac{(\log x)^3}{1 + x^2}\,dx = 0$$

since

$$\int_1^{+\infty} \frac{(\log x)^3}{1 + x^2}\,dx$$

converges, because there is K such that for $x > K$

$$\frac{(\log x)^3}{x} < 1$$

whence for $x > K$

$$\frac{(\log x)^3}{1 + x^2} < \frac{1}{1 + x^2}.$$

(c) (h,0)

We have

$$\int_0^h \frac{\cos(at)}{(1+t^2)^2}\,\mathrm{d}t = \int_0^h \frac{e^{iat}+e^{-iat}}{2(1+t^2)^2}\,\mathrm{d}t = \int_{-h}^h \frac{e^{iat}}{2(1+t^2)^2}\,\mathrm{d}t.$$

For $h > 0$ let

$$\begin{aligned}\gamma_h : [-h,h] &\to \mathbb{C}\\ t &\mapsto t\end{aligned}$$

and

$$\begin{aligned}\sigma_h : [0,\pi] &\to \mathbb{C}\\ t &\mapsto he^{it}.\end{aligned}$$

If $h > 1$ the function

$$f(z) = \frac{e^{iaz}}{(1+z^2)^2}$$

has a double pole in i inside the closed curve $\gamma_h + \delta_h$, and

$$\operatorname{res}(f,i) = -\frac{ie^{-a}(a+1)}{4}.$$

Furthermore

$$\int_{\sigma_h} f(z)\,\mathrm{d}z = \int_0^\pi \frac{e^{ihae^{it}}}{2(1+h^2e^{i2t})^2}\, ihe^{it}\,\mathrm{d}t$$

and, if $h > 1$

$$\left|\frac{e^{ihae^{it}}}{2(1+h^2e^{i2t})^2}\, ihe^{it}\right| = \frac{e^{-ha\sin t}}{2\left|1+h^2e^{i2t}\right|^2}h \le \frac{h}{2\left|h^2-1\right|^2}$$

so

$$\lim_{h\to+\infty}\int_{\sigma_h} f(z)\,\mathrm{d}z = 0$$

and

$$\int_0^{+\infty} \frac{\cos(at)}{(1+t^2)^2}\,\mathrm{d}t = 2\pi i\operatorname{res}(f,i) = \frac{\pi e^{-a}(a+1)}{2}.$$

(d) If $z = e^{i\theta}$

$$\frac{1}{a+\sin^2\theta} = -\frac{4z^2}{z^4-2(2a+1)+1}$$

so if

$$\begin{aligned}\gamma : [0,\pi] &\to \mathbb{C}\\ t &\mapsto e^{it}\end{aligned}$$

we have

$$\int_0^{\frac{\pi}{2}} \frac{1}{a+\sin^2\theta}\,d\theta = \frac{1}{4}\int_0^{2\pi} \frac{1}{a+\sin^2\theta}\,d\theta = -\frac{1}{4i}\int_\gamma \frac{4z}{z^4-2(2a+1)+1}\,dz.$$

Since $a > 0$ the integrand function f has double poles in the real points $\pm\sqrt{2a+1\pm 2\sqrt{a(a+1)}}$; now $2a+1-2\sqrt{a(a+1)} \in (0,1)$ while $2a+1+2\sqrt{a(a+1)} > 1$, so

$$\begin{aligned}\int_\gamma &\frac{4z}{z^4-2(2a+1)+1}\,dz =\\ &= 2\pi i\left(\operatorname{res}\left(f,-\sqrt{2a+1-2\sqrt{a(a+1)}}\right) + \operatorname{res}\left(f,\sqrt{2a+1-2\sqrt{a(a+1)}}\right)\right) =\\ &= 2\pi i\left(-\frac{1}{\sqrt{a(a+1)}}\right) = \left(-\frac{2\pi i}{\sqrt{a(a+1)}}\right)\end{aligned}$$

and

i

$$\int_0^{\frac{\pi}{2}} \frac{1}{a+\sin^2\theta}\,d\theta = -\frac{1}{4i}\left(-\frac{2\pi i}{\sqrt{a(a+1)}}\right) = \frac{\pi}{2\sqrt{a(a+1)}}.$$

$1/h$ h

(e)

Let

$$S = \left\{ z \in \mathbb{C} \mid -\frac{\pi}{2} < \mathfrak{Im}\, z < \frac{2\pi}{3} \right\}$$

and

$$\log^* = \exp^{-1}_{|S}$$

The function

$$f(z) = \frac{\log^*(z)}{(1+z^2)^2}$$

has double poles in $\pm i$, and

$$\mathrm{res}(f,i) = \frac{2i+\pi}{8}.$$

Let $h > 1$ and

$$\begin{aligned} \alpha_h : (0,\pi) &\to \mathbb{C} \\ t &\mapsto \frac{1}{h}\, e^{i(\pi - t)} \end{aligned}$$

$$\begin{aligned} \beta_h : (1/h, h) &\to \mathbb{C} \\ t &\mapsto t \end{aligned}$$

$$\begin{aligned} \gamma_h : (0,\pi) &\to \mathbb{C} \\ t &\mapsto h\, e^{it} \end{aligned}$$

$$\begin{aligned} \delta_h : (-h, -1/h) &\to \mathbb{C} \\ t &\mapsto t \end{aligned}$$

and $\Gamma = \alpha + \beta + \gamma + \delta$. Then

$$\int_\Gamma f(z)\,dz = \frac{-2\pi + \pi^2 i}{4}.$$

For the integral on α_h:

$$\int_{\alpha_h} f(z)\,dz = -i\int_0^{\pi} \frac{-\log h + i(\pi - t)}{\left(1 + \frac{1}{h^2}e^{2(\pi-t)i}\right)^2}\,\frac{1}{h}\, e^{2(\pi-t)i}\,dt$$

and

$$\left|\int_{\alpha_h} f(z)\,dz\right| \le \int_0^{\pi} \frac{\log h + \pi}{\left|1 + \frac{1}{h^2}e^{2(\pi-t)i}\right|^2 h}\,dt \le \int_0^{\pi} \frac{\log h + \pi}{\left|1 - \frac{1}{h^2}\right|^2 h}\,dt = \frac{(\log h + \pi)\pi}{\left|1 - \frac{1}{h^2}\right|^2 h}$$

so

$$\lim_{h\to+\infty} \int_{\alpha_h} f(z)\,dz = 0.$$

For the integral on β_h:

$$\int_{\beta_h} f(z)\,dz = \int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt.$$

For the integral on γ_h:

$$\int_{\gamma_h} f(z)\,dz = \int_0^\pi \frac{\log h + it}{(1+h^2e^{2it})^2} he^{it}\,dt$$

and

$$\left|\int_{\gamma_h} f(z)\,dz\right| \le \int_0^\pi \frac{\log h + t}{|1+h^2e^{2it}|^2} h\,dt \le \int_0^\pi \frac{\log h + \pi}{|h^2-1|^2} h\,dt \le \frac{(\log h+\pi)\pi}{|h^2-1|^2} h$$

so

$$\lim_{h\to+\infty} \int_{\gamma_h} f(z)\,dz = 0.$$

For the integral on δ_h:

$$\begin{aligned}\int_{\delta_h} f(z)\,dz &= \int_{-h}^{-1/h} \frac{\log|t| + i\pi}{(1+t^2)^2}\,dt = \int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt + \int_{1/h}^{h} \frac{i\pi}{(1+t^2)^2}\,dt = \\ &= \int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt + \frac{i\pi}{2}\left[\frac{t^2}{1+t^2} + \arctan t\right]_{1/h}^{h} = \\ &= \int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt + \frac{i\pi}{2}\left[\arctan(h) - \arctan\left(\frac{1}{h}\right)\right].\end{aligned}$$

Eventually

$$\lim_{h\to+\infty}\left(2\int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt + \frac{i\pi}{2}\left[\arctan(h) - \arctan\left(\frac{1}{h}\right)\right]\right) = \frac{-2\pi+\pi^2 i}{4}$$

and

$$\lim_{h\to+\infty}\int_{1/h}^{h} \frac{\log t}{(1+t^2)^2}\,dt = -\frac{\pi}{4}.$$

5.3 The Argument Principle

5.3.1 Exercise

5.3.2 Exercise

If $|z| = 1$

$$|f(z) - z^n + z^n| = |f(z)| < 1 \le |f(z) - z^n| + 1 = |f(z) - z^n| + |z^n|$$

so by Rouche's Theorem $V.3.8$ $f(z) - z^n$ and z^n have the same number of zeroes, counting multiplicities, in $B(0,1)$, that is, $f(z) = z^n$ has n solutions, counting multiplicities, in $B(0,1)$.

Part II

Notes

Chapter 6

The Complex Number System

6.2 The field of complex numbers

2.2 – Since

$$z = \mathfrak{Re}(z) + i\,\mathfrak{Im}(z) \tag{6.1}$$

we have

$$\overline{z} = \mathfrak{Re}(z) - i\,\mathfrak{Im}(z) \tag{6.2}$$

hence

$$\begin{aligned} z + \overline{z} &= 2\,\mathfrak{Re}(z) \\ z - \overline{z} &= 2i\,\mathfrak{Im}(z). \end{aligned}$$

2.3 – We have

$$\begin{aligned} \overline{z+w} &= \overline{\mathfrak{Re}(z) + i\,\mathfrak{Im}(z) + \mathfrak{Re}(w) + i\,\mathfrak{Im}(w)} \\ &= \overline{(\mathfrak{Re}(z) + \mathfrak{Re}(w)) + i(\mathfrak{Im}(z) + \mathfrak{Im}(w))} \\ &= (\mathfrak{Re}(z) + \mathfrak{Re}(w)) - i(\mathfrak{Im}(z) + \mathfrak{Im}(w)) \\ &= (\mathfrak{Re}(z) - i\,\mathfrak{Im}(z)) + (\mathfrak{Re}(w) - i\,\mathfrak{Im}(w)) \\ &= \overline{z} + \overline{w} \end{aligned}$$

and

$$\begin{aligned} \overline{zw} &= \overline{(\mathfrak{Re}(z)\,\mathfrak{Re}(w) - \mathfrak{Im}(z)\,\mathfrak{Im}(w)) + i(\mathfrak{Re}(z)\,\mathfrak{Im}(w) + \mathfrak{Im}(z)\,\mathfrak{Re}(w))} \\ &= (\mathfrak{Re}(z)\,\mathfrak{Re}(w) - \mathfrak{Im}(z)\,\mathfrak{Im}(w)) - i(\mathfrak{Re}(z)\,\mathfrak{Im}(w) + \mathfrak{Im}(z)\,\mathfrak{Re}(w)) \\ &= (\mathfrak{Re}(z) - i\,\mathfrak{Im}(z))(\mathfrak{Re}(w) - i\,\mathfrak{Im}(w)) \\ &= \overline{z}\,\overline{w}. \end{aligned}$$

2.4 – We have

$$\begin{aligned} |zw|^2 &= (zw)(\overline{zw}) \\ &= (zw)(\overline{z}\,\overline{w}) \\ &= (z\overline{z})(w\overline{w}) \\ &= |z|^2|w|^2 \\ &= (|z||w|)^2 \end{aligned}$$

hence

$$|zw| = |z||w|.$$

2.5 – We have

$$\left|\frac{1}{z}\right| |z| = \left|\frac{1}{z}\, z\right| = |1| = 1 \tag{6.3}$$

hence

$$\left|\frac{1}{z}\right| = \frac{1}{|z|} \tag{6.4}$$

and

$$\left|\frac{z}{w}\right| = |z| \left|\frac{1}{w}\right| = \frac{|z|}{|w|}. \tag{6.5}$$

2.6 – We have

$$|\overline{z}|^2 = \overline{z}\,\overline{(\overline{z})} = \overline{z}z = |z|^2 \tag{6.6}$$

hence

$$|\overline{z}| = |z|\,. \tag{6.7}$$

Chapter 7

Metric Spaces and the Topology of $\mathbb{C}$

7.2 Connectedness

Proof of Proposition **2.8**

Proof.

(a) If B is not connected, then it has more than one component. Since A is connected, there must be a component C_A such that $A \subseteq C_A$. Call C another component of B and take $x \in C$. There are two open subsets H_1 and H_2 of X such that $C_A \subseteq H_1$, $C \subseteq H_2$ and $H_1 \cap H_2 = \emptyset$. Then there is an open ball $B(x,r)$ such that $B(x,r) \subseteq H_2$ which yields $B(x,r) \cap A = \emptyset$, so that $x \notin \overline{A}$.

(b) If C is a component of X, $C \subseteq \overline{C}$ always holds. But for point **(a)** $\overline{C}$ also is connected, so $\overline{C} \subseteq C$.

□

Chapter 8

Elementary Properties and Examples of Analytic Functions

8.3 Analitic functions as mappings. Möbius transformations

Proposition 8.3.1. *The linear fractional transformation S defined by*

$$S(z) = \frac{az+b}{cz+d} \tag{8.1}$$

is invertible if and only if $ad - bc \neq 0$ and constant if and only if $ad - bc = 0$.

Proof. If $ad - bc \neq 0$ let T be the linear fractional transformation defined by

$$T(w) = \frac{dw-b}{-cw+a}. \tag{8.2}$$

$$T(S(w)) = \frac{a\frac{dw-b}{-cw+a}+b}{c\frac{dw-b}{-cw+a}+d} = \frac{\frac{adw-ab-cbw+ab}{-cw+a}}{\frac{cdw-cb-cdw+ad}{-cw+a}} = \frac{(ad-bc)w}{ad-bc} = w \tag{8.3}$$

and

$$S(T(z)) = \frac{d\frac{az+b}{cz+d}-b}{-c\frac{az+b}{cz+d}+a} = \frac{\frac{adz+bd-bcz-bd}{cz+d}}{\frac{-acz-bc+acz+ad}{cz+d}} = \frac{(ad-bc)z}{ad-bc} = z. \tag{8.4}$$

If $ad - bc = 0$ then, if $d \neq 0$

$$S(z) = \frac{\frac{bc}{d}z+b}{cz+d} = \frac{bcz+bd}{cdz+d^2} = \frac{b(cz+d)}{d(cz+d)} = \frac{b}{d}; \tag{8.5}$$

if $d = 0$ then $bc = 0$ which implies, since it cannot be $d = c = 0$, that $b = 0$, and

$$S(z) = \frac{az}{cz} = \frac{a}{c}. \tag{8.6}$$

□

Proposition 8.3.2. *The Möbius transformation T defined by*

$$T(z) = \frac{az+b}{cz+d} \tag{8.7}$$

satisfies $T^2 = I$ and $T \neq I$ if and only if $a = -d$.

Proof. By computation

$$T^2(z) = \frac{(a^2+bc)z+ab+bd}{(ca+cd)z+cb+d^2} \tag{8.8}$$

so $T^2(z) = z$ if and only if

$$c(a+d)z^2 + (a+d)(d-a)z - (a+d)b = 0. \tag{8.9}$$

This holds for every $z \in \mathbb{C}$ if and only if $a = -d$. □

Proposition 8.3.3. *If T is a Möbius transformation and $T^2 = I$ then T has two distinct fixed points.*

Proof. By Proposition 8.3.2 if $T^2 = I$ then

$$T(z) = \frac{az+b}{cz-a}. \tag{8.10}$$

Then $T(z) = z$ if and only if.

$$cz^2 - 2az - b = 0 \tag{8.11}$$

This equation has two different roots if and only if

$$a^2 + bc \neq 0 \tag{8.12}$$

which holds if T is a Möbius transformation. □

Proposition 8.3.4. *If z_1, z_2 are two distinct points in $\mathbb{C}$, there is only one Möbius transformation T such that $T^2 = I$ and z_1, z_2 are the fixed points of T, and it is represented by*

$$T(z) = \frac{(z_1+z_2)z - 2z_1z_2}{2z - (z_1+z_2)} \tag{8.13}$$

Proof. Let

$$T(z) = \frac{az+b}{cz-a}. \tag{8.14}$$

Then $T(z) = z$ if and only if

$$cz^2 - 2az - b = 0. \tag{8.15}$$

If θ_1 and θ_2 are the two square roots of $a^2 + bc$, the last equation has the two roots

$$z_1 = \frac{a+\theta_1}{c} \tag{8.16}$$

$$z_2 = \frac{a+\theta_2}{c}. \tag{8.17}$$

We can suppose $a^2 + bc = 1$, otherwise we can divide every coefficient in the representation of T by any τ such that $\tau^2 = a^2 + bc$, and get another representation of T. Then

$$z_1 = \frac{a+1}{c} \tag{8.18}$$

$$z_2 = \frac{a-1}{c}. \tag{8.19}$$

Hence

$$a = \frac{z_1+z_2}{z_1-z_2} \tag{8.20}$$

$$c = \frac{2}{z_1-z_2}. \tag{8.21}$$

From $a^2 + bc = 1$ follows

$$b = \frac{2z_1z_2}{z_2-z_1}. \tag{8.22}$$

It is a simple check to verify that if T is represented by (8.13) then z_1 and z_2 are its fixed points. □

Proposition 8.3.5. *Let T, S be Möbius transformations such that $T^2 = S^2 = I$, with fixed points respectively z_1, z_2 and w_1, w_2. If $T(w_1) = w_2$ then $S(z_1) = z_2$.*

Proof. We know that the Möbius transformation S that has fixed points w_1 and w_2 is represented by

$$S(z) = \frac{(w_1 + w_2)z - 2w_1w_2}{2z - (w_1 + w_2)}. \tag{8.23}$$

If T has fixed points z_1 and z_2, and $w_2 = T(w_1)$, then

$$w_2 = \frac{(z_1 + z_2)w_1 - 2z_1z_2}{2w_1 - (z_1 + z_2)}. \tag{8.24}$$

Therefore

$$S(z) = \frac{\left(w_1 + \frac{(z_1+z_2)w_1 - 2z_1z_2}{2w_1-(z_1+z_2)}\right) z - 2w_1 \frac{(z_1+z_2)w_1-2z_1z_2}{2w_1-(z_1+z_2)}}{2z - \left(w_1 + \frac{(z_1+z_2)w_1-2z_1z_2}{2w_1-(z_1+z_2)}\right)} = \tag{8.25}$$

$$= \frac{(2w_1^2 - 2z_1z_2)z - 2w_1^2(z_1 + z_2) + 4w_1z_1z_2}{(4w_1 - 2(z_1 + z_2))z - (2w_1^2 - 2z_1z_2)} \tag{8.26}$$

$$\tag{8.27}$$

and

$$S(z_1) = \frac{w_1^2z_1 - z_1^2z_2 - w_1^2z_1 - w_1^2z_2 + 2w_1z_1z_2}{2w_1z_1 - z_1^2 - z_1z_2 - w_1^2 + z_1z_2} = \tag{8.28}$$

$$= \frac{-z_1^2z_2 - w_1^2z_2 + 2w_1z_1z_2}{2w_1z_1 - z_1^2 - w_1^2} = \tag{8.29}$$

$$= \frac{(z_1 - w_1)^2z_2}{(z_1 - w_1)^2} = z_2. \tag{8.30}$$

□

Proposition 8.3.6. *If T, S are Möbius transformations such that $T^2 = S^2 = I$ and T swaps S's fixed points, then $ST = TS$.*

Proof. Let z_1, z_2 and w_1, w_2 be the fixed points of T and S. Then $T(w_1) = w_2$ and by Proposition 8.3.5 $S(z_1) = z_2$. Let R be the Möbius transformation defined by

$$\begin{cases} R(0) = z_1 \\ R(\infty) = z_2 \\ R(1) = w_1. \end{cases} \tag{8.31}$$

Then

$$\begin{cases} R^{-1}SR(0) = \infty \\ R^{-1}SR(\infty) = 0 \\ R^{-1}SR(1) = 1 \end{cases} \tag{8.32}$$

which implies that $R^{-1}SR(z) = \frac{1}{z}$. Furthermore

$$\begin{cases} R^{-1}TR(0) = 0 \\ R^{-1}TR(\infty) = \infty \\ R^{-1}TR(1) = R^{-1}(w_2) \end{cases} \tag{8.33}$$

but $R^{-1}SR(R^{-1}(w_2)) = R^{-1}S(w_2) = R^{-1}(w_2)$ whence $R^{-1}(w_2) = 1$ or $R^{-1}(w_2) = -1$, but $R^{-1}(w_1) = 1$, so $R^{-1}(w_1) = -1$. Then $R^{-1}TR(z) = -z$, and $R^{-1}TR$ and $R^{-1}SR$ commute. But then $R^{-1}TRR^{-1}SR = R^{-1}SRR^{-1}TR$ implies $TS = ST$. □

Proposition 8.3.7. *If T is a Möbius transformation such that $T^2 = I$ with fixed points z_1 and z_2, then for any $z \in \mathbb{C}$ the points $z, T(z), z_1, z_2$ lie on a circle.*

Proof. We have

$$(T(z), z_1, z_2, z) = (z, z_1, z_2, T(z)), \tag{8.34}$$

that is

$$\frac{z - z_2}{z - T(z)} \frac{z_1 - T(z)}{z_1 - z_2} = \frac{T(z) - z_2}{T(z) - z} \frac{z_1 - z}{z_1 - z_2} \tag{8.35}$$

whence

$$\frac{\frac{z-z_2}{T(z)-z_2}}{\frac{z-z_1}{T(z)-z_1}} = -1 \tag{8.36}$$

that is

$$(z, T(z), z_2, z_1) = -1. \tag{8.37}$$

□

Chapter 9

Complex Integration

9.1 Riemann-Stieltjes integrals

Proposition 9.1.1. *Let* $\gamma : [a, b] \to \mathbb{C}$ *be a rectifiable path,* $a < h < b$*. Define*

$$\gamma_1 = \gamma_{|[a,h]}$$
$$\gamma_2 = \gamma_{|[h,b]}.$$

Then $V(\gamma) = V(\gamma_1) + V(\gamma_2)$*.*

Proof. Let

$$\mathcal{P}^1 = \{t_0, t_1, \dots, t_n\}$$
$$\mathcal{P}^2 = \{s_0, s_1, \dots, s_m\}$$

be partitions of $[a, h]$ and $[h, b]$. Then $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2$ is a partition of $[a, b]$ and

$$V(\gamma_1, \mathcal{P}^1) + V(\gamma_2, \mathcal{P}^2) = V(\gamma, \mathcal{P}) \leq V(\gamma).$$

whence

$$V(\gamma_1) + V(\gamma_2) \leq V(\gamma).$$

If

$$\mathcal{P} = \{t_0, t_1, \dots, t_n\}$$

is a partition of $[a, b]$, let k be such that $h \in [t_k, t_{k+1}]$. Then

$$\mathcal{P}' = \{t_0, t_1, \dots, t_k, h, t_{k+1}, \dots, t_n\}$$

is a refinement of $\mathcal{P}$ and

$$\mathcal{P}^1 = \{t_0, t_1, \dots, t_k, h\}$$
$$\mathcal{P}^2 = \{h, t_{k+1}, \dots, t_n, \}$$

are partitions of $[a, h]$ and $[h, b]$. Then

$$V(\gamma, \mathcal{P}) = V(\gamma_1, \mathcal{P}^1) + V(\gamma_2, \mathcal{P}^2) \leq V(\gamma_1) + V(\gamma_2)$$

whence

$$V(\gamma) \leq V(\gamma_1) + V(\gamma_2).$$

□

Proposition 9.1.2. *Let* $\gamma : [a, b] \to \mathbb{C}$ *be a contionuous path, and suppose that for every* h *such that* $a < h < b$ *the path* $\gamma_h = \gamma_{|[h,b]}$ *is rectifiable, and*

$$I = \lim_{h \to a^+} V(\gamma_h) \in \mathbb{R}. \tag{9.1}$$

Then γ *is rectifiable, and* $V(\gamma) = I$*.*

Proof. Let $\epsilon > 0$, and let $\mathcal{P}$ be any partition of $[a,b]$. Take δ such that for $x_1, x_2 \in [a,b]$, $|x_1 - x_2| < \delta$ yields $|\gamma(x_1) - \gamma(x_2)| < \epsilon$ (of course γ is uniformly continuous). Now, take a refinement of $\mathcal{P}$

$$\mathcal{P}' = \{t_0, t_1, \ldots, t_n\} \tag{9.2}$$

such that $|\mathcal{P}'| < \delta$. Then

$$\begin{aligned} v(\gamma, \mathcal{P}) \leq v(\gamma, \mathcal{P}') = |\gamma(t_1) - \gamma(t_0)| + \sum_{k=2}^{n} |\gamma(t_k) - \gamma(t_{k-1})| < \\ < \epsilon + \sum_{k=2}^{n} |\gamma(t_k) - \gamma(t_{k-1})| \leq \\ < \epsilon + V\left(\gamma_{|[t_1,b]}\right) \leq \epsilon + I. \end{aligned}$$

Hence for every $\epsilon > 0$ $V(\gamma) \leq \epsilon + I$, which yields $V(\gamma) \leq I$.

On the other hand, if $a < h < b$ and $\mathcal{P}$ is any partition of $[h,b]$, we have

$$v(\gamma_h, \mathcal{P}) \leq V(\gamma) \tag{9.3}$$

which yields

$$V(\gamma_h) \leq V(\gamma) \tag{9.4}$$

and eventually

$$I \leq V(\gamma). \tag{9.5}$$

□

Corollary 9.1.1. *Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path and $a < t < b$. Define $\gamma_t = \gamma_{|[a,t]}$. Then*

$$\lim_{t \to b^-} V(\gamma_t) = V(\gamma).$$

Proof. The function

$$\begin{aligned} f : [a,b) &\to \mathbb{R}^+ \\ t &\mapsto V(\gamma_t) \end{aligned}$$

is increasing, and $f(t) \leq V(\gamma)$, so its limit in b^- exists in $\mathbb{R}$. By Proposition 9.1.2 this limit is $V(\gamma)$. □

Corollary 9.1.2. *Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path and $a < t < b$. Define $\sigma_t = \gamma_{|[t,b]}$. Then*

$$\lim_{t \to b^-} V(\sigma_t) = 0.$$

Proof. Let $\gamma_t = \gamma_{|[a,t]}$. Then $V(\gamma_t) + V(\sigma_t) = V(\gamma)$ so by Corollary 9.1.1

$$\lim_{t \to b^-} V(\sigma_t) = V(\gamma) - \lim_{t \to b^-} V(\sigma_t) = 0.$$

□

Proposition 9.1.3. *Let t_k be a sequence of real numbers such that $t_k \in [a,b]$, $t_0 = a$, $t_{k+1} > t_k$ and $\lim t_k = b$, and for each k let $\gamma_k : [t_k, t_{k+1}] \to \mathbb{C}$ be a smooth path, with $\gamma_k(t_{k+1}) = \gamma_{k+1}(t_{k+1})$, and suppose that the series*

$$\sum_{k=0}^{+\infty} V(\gamma_k)$$

converges. Then $\gamma_k(t_k)$ converges, the path $\gamma : [a,b] \to \mathbb{C}$ defined by

$$\gamma(t) = \begin{cases} \gamma_k(t) & \text{if } t \in [t_k, t_{k+1}) \quad k \in \mathbb{N} \\ \lim \gamma_k(t_k) & \text{if } t = b \end{cases}$$

is rectifiable, and

$$V(\gamma) = \sum_{k=0}^{+\infty} V(\gamma_k).$$

Furthermore if f is a continuous function defined on an open subset of $\mathbb{C}$ contaning $\{\gamma\}$

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{+\infty} \int_{\gamma_k} f(z)\,dz$$

Proof. If $\epsilon > 0$ there is $\bar{n} \in \mathbb{N}$ such that $n_2 > n_1 \geq \bar{n}$ implies

$$\sum_{k=n_1}^{n_2} V(\gamma_k) < \epsilon.$$

So if $n_2 > n_1 \geq \bar{n}$

$$\begin{aligned} |\gamma_{n_2}(t_{n_2}) - \gamma_{n_1}(t_{n_1})| &= \left| \sum_{k=n_1}^{n_2-1} \gamma_{k+1}(t_{k+1}) - \gamma_k(t_k) \right| = \\ &= \left| \sum_{k=n_1}^{n_2-1} \gamma_k(t_{k+1}) - \gamma_k(t_k) \right| \leq \\ &\leq \sum_{k=n_1}^{n_2-1} |\gamma_k(t_{k+1}) - \gamma_k(t_k)| \leq \\ &\leq \sum_{k=n_1}^{n_2-1} V(\gamma_k) < \epsilon \end{aligned}$$

which means that $\gamma_k(t_k)$ is a Cauchy sequence, and thus it converges. Let

$$l = \lim \gamma_k(t_k).$$

Now if $\epsilon > 0$, let $\bar{k}$ such that $k \geq \bar{k}$ implies $|l - \gamma_k(t_k)| < \frac{\epsilon}{2}$ and $V(\gamma_k) < \frac{\epsilon}{2}$. If $\delta = b - t_{\bar{k}}$ then $b - t < \delta$ implies $t \in [t_{k_0}, t_{k_0+1}]$ for some $k_0 \geq \bar{k}$, thus if $b - t < \delta$ then

$$|\gamma(b) - \gamma(t)| = |l - \gamma_{k_0}(t_{k_0}) + \gamma_{k_0}(t_{k_0}) - \gamma(t)| < \frac{\epsilon}{2} + V(\gamma_{k_0}) < \epsilon.$$

This proves that γ is continuous in b.

Now, if $t \in [a, b)$ the path $\gamma_t = \gamma_{|[a,t]}$ is piecewise smooth, so if $\bar{n}$ is such that $t \leq t_{\bar{n}}$ then

$$V(\gamma_t) \leq \sum_{k=0}^{\bar{n}} V(\gamma_k) \leq \sum_{k=0}^{\infty} V(\gamma_k)$$

so

$$V(\gamma) \leq \sum_{k=0}^{\infty} V(\gamma_k).$$

If $\epsilon > 0$ there is $\bar{n}$ such that

$$\sum_{k=0}^{\bar{n}} V(\gamma_k) > \sum_{k=0}^{\infty} V(\gamma_k) - \epsilon$$

thus

$$V(\gamma) \geq V(\gamma_{|[a,t_{\bar{n}+1}]}) = \sum_{k=0}^{\bar{n}} V(\gamma_k) > \sum_{k=0}^{\infty} V(\gamma_k) - \epsilon.$$

Since this holds for any $\epsilon > 0$, it yields

$$V(\gamma) \geq \sum_{k=0}^{\infty} V(\gamma_k).$$

If f is a continuous function defined on an open subset of $\mathbb{C}$ contaning $\{\gamma\}$, let $\bar{\gamma}_k = \gamma_{|[a,t_k]}$. Then

$$\int_\gamma f(z)\,dz = \lim_{k\to+\infty} \int_{\bar{\gamma}_k} f(z)\,dz = \lim_{k\to+\infty} \sum_{h=0}^{k-1} \int_{\gamma_h} f(z)\,dz = \sum_{h=0}^{+\infty} \int_{\gamma_h} f(z)\,dz.$$

□

Appendix A

Miscellaneous

A.1 Identity 1

The following identity holds (in any ring):

$$\sum_{k=0}^{n}\sum_{h=0}^{k} a_h b_{k-h} = \sum_{k=0}^{n}\sum_{h=0}^{n-k} a_k b_h. \tag{A.1}$$

Proof. By induction on n. If $n = 0$ then the left side is

$$\sum_{k=0}^{0}\sum_{h=0}^{k} a_h b_{k-h} = \sum_{h=0}^{0} a_h b_{0-h} = a_0 b_0 \tag{A.2}$$

and the right one

$$\sum_{k=0}^{0}\sum_{h=0}^{0-k} a_k b_h = \sum_{h=0}^{0} a_0 b_h = a_0 b_0. \tag{A.3}$$

Now, supposing the identity holds for n:

$$\begin{aligned}
\sum_{k=0}^{n+1}\sum_{h=0}^{k} a_h b_{k-h} &= \sum_{k=0}^{n}\sum_{h=0}^{k} a_h b_{k-h} + \sum_{h=0}^{n+1} a_h b_{n+1-h} = \\
&= \sum_{k=0}^{n}\sum_{h=0}^{n-k} a_k b_h + \sum_{h=0}^{n+1} a_h b_{n+1-h} = \\
&= \sum_{k=0}^{n}\sum_{h=0}^{n-k} a_k b_h + \sum_{h=0}^{n} a_h b_{n+1-h} + a_{n+1} b_0 = \\
&= \sum_{k=0}^{n}\sum_{h=0}^{n-k} a_k b_h + \sum_{k=0}^{n} a_k b_{n+1-k} + a_{n+1} b_0 = \\
&= \sum_{k=0}^{n}\sum_{h=0}^{n+1-k} a_k b_h + \sum_{h=0}^{0} a_{n+1} b_h = \\
&= \sum_{k=0}^{n}\sum_{h=0}^{n+1-k} a_k b_h + \sum_{h=0}^{n+1-(n+1)} a_{n+1} b_h \\
&= \sum_{k=0}^{n+1}\sum_{h=0}^{n+1-k} a_k b_h.
\end{aligned}$$

□

A.2 The real and immaginary part and module of sin and cos

$$\begin{aligned}\sin(x+iy) &= \frac{e^{i(x+iy)}-e^{-i(x+iy)}}{2i} = \frac{e^{ix-y}-e^{-ix+y}}{2i} = \\ &= \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} = \\ &= \frac{-\cos x(e^{y}-e^{-y}) + i\sin x(e^{y}+e^{-y})}{2i} = \\ &= \sin x\cosh y + i\cos x\sinh y.\end{aligned}$$

$$\begin{aligned}|\sin(x+iy)|^2 &= (\sin x)^2(\cosh y)^2 + (\cos x)^2(\sinh y)^2 = \\ &= (\sin x)^2 + (\sin x)^2(\sinh y)^2 + (\cos x)^2(\sinh y)^2 = \\ &= (\sin x)^2 + (\sinh y)^2.\end{aligned}$$

$$\begin{aligned}\cos(x+iy) &= \frac{e^{i(x+iy)}+e^{-i(x+iy)}}{2} = \frac{e^{ix-y}+e^{-ix+y}}{2} = \\ &= \frac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2} = \\ &= \frac{\cos x(e^{y}+e^{-y}) - i\sin x(e^{y}-e^{-y})}{2} = \\ &= \cos x\cosh y - i\sin x\sinh y.\end{aligned}$$

$$\begin{aligned}|\cos(x+iy)|^2 &= (\cos x)^2(\cosh y)^2 + (\sin x)^2(\sinh y)^2 = \\ &= (\cos x)^2 + (\cos x)^2(\sinh y)^2 + (\sin x)^2(\sinh y)^2 = \\ &= (\cos x)^2 + (\sinh y)^2.\end{aligned}$$

A.3 The group $SL_2(\mathbb{C})$

Proposition A.3.1. *The group $SL_2(\mathbb{C})$ has no normal non trivial subgroup other than*

$$\mathcal{K} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}. \tag{A.4}$$

Proof. Clearly $\mathcal{K}$ is a normal subgroup of $SL_2(\mathbb{C})$. Let $\mathcal{N}$ be a normal subgroup of $SL_2(\mathbb{C})$ and $\mathcal{N} \neq \mathcal{K}$, $\mathcal{N} \neq \{I\}$. We will prove that $\mathcal{N} = SL_2(\mathbb{C})$ by showing that for any conjugated class $\mathcal{C}$ of $SL_2(\mathbb{C})$ there is $L \in \mathcal{N} \cap \mathcal{C}$. Since all the elements of $SL_2(\mathbb{C})$ belonging to the same conjugated class must also be in $\mathcal{N}$, this proves that $\mathcal{N} = SL_2(\mathbb{C})$.

In $SL_2(\mathbb{C})$ there is, for any λ_1, λ_2 such that $\lambda_1 \neq \lambda_2$, one conjugated class containing

$$D_{\lambda_1,\lambda_2} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

one conjugated class containing only

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

one conjugated class containing only

$$-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

one conjugated class containing

$$H = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

one conjugated class containing

$$K = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Let $A \in \mathcal{N}$ such that $A \neq I$. Let $\delta_1, \delta_2 \in \mathbb{C}$ and $h = \delta_1 + \delta_2$. Now we show that there is $Y \in \mathcal{N}$ such that the eigenvalues of Y are δ_1, δ_2.

If the with eigenvalues of A are λ_1 and λ_2 such that $\lambda_1 \neq \lambda_2$, then all the elements of $SL_2(\mathbb{C})$ with the same eigenvalues must be in $\mathcal{N}$, in particular

$$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \in \mathcal{N}. \tag{A.5}$$

Let

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}. \tag{A.6}$$

The linear system

$$\begin{cases} \lambda_1 x + \lambda_2 t = h \\ x + t = \lambda_1 + \lambda_2 \end{cases} \tag{A.7}$$

has one solution, for any h, λ_1, λ_2, since $\lambda_1 \neq \lambda_2$, and that means $\mathrm{Tr}(XB) = h$, $\mathrm{Tr}(X) = \lambda_1 + \lambda_2$. Choosing z, y such that $xt - zy = 1$ we have $X \in \mathcal{N}$, so $XB \in \mathcal{N}$ and $\mathrm{Tr}(XB) = h$, which implies that the eigenvalues of XB are δ_1, δ_2. If $h \neq 2$ and $h \neq -2$ then $\delta_1 \neq \delta_2$. If $h = 2$ then $\delta_1 = \delta_2 = 1$ and $XB \neq I$, since B^{-1} has the same eigenvalues as B, so XB is similar to H. If $h = -2$ then $\delta_1 = \delta_2 = -1$ and $XB \neq -I$, since $-B^{-1}$ has the same eigenvalues as $-B$, so XB is similar to K. Finally, if C is similar to H and D is similar to K then CD is similar to $HK = -I$.

If the eigenvalues of A are not distinct, A must not be similar either to I or $-I$, so it is similar to H or to K.

Suppose A similar to H. Let

$$X = \begin{pmatrix} x & y \\ z & t \end{pmatrix}. \tag{A.8}$$

The linear system

$$\begin{cases} x + z + t = h \\ x + t = 2 \end{cases} \tag{A.9}$$

has ∞^1 solutions, for any h, for all of which $z = h - 2$. If $h \neq 2$ we can choose y such that $xt - zy = 1$, and XH is similar either to D_{δ_1, δ_2} if $h \neq -2$, or to K if $h = -2$. If $h = 2$ there's no need to show anything since any $Z \in SL_2(\mathbb{C})$ such that $\mathrm{Tr}(Z) = 2$ is already similar to A.

The same holds if A is similar to K. □